Problem 1 Chapter 14.4

Determine the maximum/minimum value of $f(x,y) = x^2 + xy + y^2$ on the unit disk $D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\}$.

I had a typo here!

Solution Since f is continuous and D is closed and bounded, f has max/min in D.

① Find critical points that satisfy $x^2+y^2 < 1$
So set

$$0 = f_x = 2x+y \qquad \Rightarrow -4y+y=0$$
$$0 = f_y = 2y+x \qquad \Rightarrow y=0$$

Only critical point in this case is $(0,0)$.

② Find critical points for which $x^2+y^2=1$

By Lagrange: $\exists \lambda \in \mathbb{R}$ such that $\nabla f = \lambda \nabla g$ where $g = x^2+y^2$

$$(2x+y, 2y+x) = \nabla f = \lambda \nabla g = (2x, 2y)$$

$$\Rightarrow \quad 2x+y = 2\lambda x$$
$$2y+x = 2\lambda y$$

Case 1 If $x=0$, then $y=0$. Also, if $y=0$, then $x=0$.

Case 2 Assume $x,y \neq 0$ so we can divide. then

$$2+\frac{y}{x} = \frac{2x+y}{x} = 2\lambda = \frac{2y+x}{y} = 2+\frac{x}{y}$$

$$\Rightarrow \frac{y}{x} = \frac{x}{y} \Rightarrow y^2 = x^2$$

By constraint $x^2+y^2=1$ we see that $x = \pm\frac{\sqrt{2}}{2}$, $y = \pm\frac{\sqrt{2}}{2}$
(4 critical points)

Evaluate $f(x,y)$ at all critical pts to see that $\max(f) = \frac{3}{2}$ and $\min(f) = 0$ ∎

Problem 2

Chapter 14.4

Find the absolute maximum/minimum of $f(x,y,z) = 2x+y$ subject to the constraint $x+y+z=1$.

Solution Note that $x+y+z=1$ is closed but unbounded so it is not guaranteed that max/min exists. Solve for y:

$y = 1-x-z$ so that

$$f(x,y,z) = 2x+1-x-z$$
$$= x-z+1$$

Then $f \to \infty$ if $x \to \infty$ w/ $z=0$ $\Rightarrow$ f has no max

$f \to -\infty$ if $z \to \infty$ w/ $x=0$ $\Rightarrow$ f has no min

Another way: Lagrange multipliers $\Rightarrow$ $\exists \lambda \in \mathbb{R}$ such that

$$(2,1,0) = \nabla f = \lambda \nabla g = \lambda(1,1,1)$$

$$\Rightarrow \begin{matrix} 2=\lambda \\ 1=\lambda \\ 0=\lambda \end{matrix}$$ $\Rightarrow$ contradiction so there are no critical pts. $\blacksquare$

Problem 3 | Chapter 14.7

Let $F(x,y,z) = (e^{xz}, \sin(xy), x^5y^3z^2)$. Compute the divergence and curl of F.

Solution

Divergence: $\nabla \cdot F$ where $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$

Curl: $\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$ (say $F = (F_1, F_2, F_3)$)

Divergence

$$\nabla \cdot F = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \cdot (e^{xz}, \sin xy, x^5y^3z^2)$$
$$= \partial/\partial x(e^{xz}) + \partial/\partial y(\sin xy) + \partial/\partial z(x^5y^3z^2)$$
$$= ze^{xz} + x\cos(xy) + 2x^5y^3z.$$

Curl

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{xz} & \sin xy & x^5y^3z^2 \end{vmatrix}$$

$$= i\left(\partial/\partial y(x^5y^3z^2) - \partial/\partial z(\sin xy)\right) - j\left(\partial/\partial x(x^5y^3z^2) - \partial/\partial z(e^{xz})\right)$$
$$+ k\left(\partial/\partial x(\sin xy) - \partial/\partial y(e^{xz})\right)$$

$$= \left(3x^5y^2z^2, \; xe^{xz} - 5x^4y^3z^2, \; y\cos(xy)\right)$$

$\square$

Problem 4

Chapter 14.7

Suppose $F:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a C^2 vector field. Show that

$$\operatorname{div}(\operatorname{curl} F) = 0.$$

(Memorize this thm)

Proof
$$\begin{aligned}\operatorname{div}(\operatorname{curl} F) &= \nabla \cdot \operatorname{curl} F \qquad (\text{Say } F = (F_1, F_2, F_3))\\ &= \nabla \cdot (\nabla \times F)\\ &= (\partial/\partial x, \partial/\partial y, \partial/\partial z) \cdot \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}\end{aligned}$$

$$= (\partial/\partial x, \partial/\partial y, \partial/\partial z) \cdot (\partial/\partial y F_3 - \partial/\partial z F_2, \partial/\partial z F_1 - \partial/\partial x F_3, \partial/\partial x F_2 - \partial/\partial y F_1)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} F_3 - \partial/\partial z F_2\right) + \partial/\partial y(\partial/\partial z F_1 - \partial/\partial x F_3) + \partial/\partial z(\partial/\partial x F_2 - \partial/\partial y F_1)$$

$$= \cancel{\frac{\partial}{\partial x \partial y} F_3} - \cancel{\frac{\partial}{\partial x \partial z} F_2} + \cancel{\frac{\partial}{\partial y \partial z} F_1} - \cancel{\frac{\partial}{\partial y \partial x} F_3} + \cancel{\frac{\partial}{\partial z \partial x} F_2} - \cancel{\frac{\partial}{\partial z \partial y} F_1} = 0.$$

By assumption $F = (F_1, F_2, F_3)$ is C^2 so F_1, F_2, F_3 are C^2.

So the mixed Partials are equal by Clairaut's Thm, e.g. $\frac{\partial}{\partial z \partial y} F_1 = \frac{\partial}{\partial y \partial z} F_1$

$\blacksquare$

Problem 5 — Chapter 14.7

Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a C^2 function. Then show that $\text{curl}(\nabla f) = \vec{0}$. (memorize this thm)
Is the vector field $F(x,y,z) = (2x-5y)\,i + (4x+y)\,j$ a gradient field?

Proof $\text{Curl}(\nabla f) = \nabla \times (\nabla f)$ $\quad (\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z))$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \qquad \left(\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\right)$$

$$= i\left(\frac{\partial f}{\partial y \partial z} - \frac{\partial f}{\partial y \partial z}\right) - j\left(\frac{\partial f}{\partial x \partial z} - \frac{\partial f}{\partial z \partial x}\right) + k\left(\frac{\partial f}{\partial x \partial y} - \frac{\partial f}{\partial y \partial x}\right)$$

$= (0,0,0)$ by Clairaut's Thm again since f is C^2.

A vector field F is called a gradient field if there is a C^2 function $f: \mathbb{R}^3 \to \mathbb{R}$ such that $F = \nabla f$.

Suppose $F(x,y,z) = (2x-5y)\,i + (4x+y)\,j$ is a gradient field, then there is a C^2 function f such that $F = \nabla f$. By the Thm we proved we have

$$\vec{0} \overset{\downarrow}{=} \text{curl}(\nabla f) = \text{curl}(F)$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-5y & 4x+y & 0 \end{vmatrix}$$

$$= i\left(\partial/\partial y(0) - \partial/\partial z(4x+y)\right) - j\left(\partial/\partial x(0) - \partial/\partial z(2x-5y)\right) + k\left(\partial/\partial x(4x+y) - \partial/\partial y(2x-5y)\right)$$

$$= (0, 0, 4+5)$$

So F is not a gradient vector field. (Compare w/ Problem 3 from Friday week 4 posted on my website)

Problem 6 | Chapter 14.4

Determine the maximum/minimum value of $f(x,y) = x + yz$ on the unit disk $D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2+z^2 \le 1\}$

Solution

(1) Find all critical points for which $x^2+y^2<1$:

Solve:
$$0 = f_x = 1$$
$$0 = f_y = z$$
$$0 = f_z = y$$
$\Rightarrow$ There are no critical points where $x^2+y^2<1$

(2) Find all critical points for which $x^2+y^2=1$:

Solve $\nabla f = \lambda \nabla g$, $g = x^2+y^2+z^2$

$\Rightarrow (1, z, y) = \lambda(2x, 2y, 2z)$

$\Rightarrow$
(1) $2\lambda x = 1$
(2) $2\lambda y = z$
(3) $2\lambda z = y$
(4) $x^2+y^2+z^2=1$

$\boxed{x \neq 0 \text{ by } (1)}$

Solve (1): $2\lambda = \frac{1}{x}$ $\Rightarrow$ Substitute into (2) and (3)

$\Rightarrow$ (2) $\frac{y}{x} = z$ (3) $\frac{z}{x} = y$

$\Rightarrow$ (2) $y = xz$ (3) $z = xy$

$\Rightarrow$ (5) $\boxed{y = x^2 y}$

Case 1 $y=0$, $\Rightarrow z=0 \Rightarrow$ (4) $x^2=1 \Rightarrow x=\pm 1$

$(\pm 1, 0, 0)$ are critical pts

Case 2 $y \neq 0$, By (5), $x^2 = \frac{y}{y} = 1 \Rightarrow x = \pm 1$.

So by (1) $\lambda = \pm\frac{1}{2}$ $\Rightarrow$ (2) $z = \pm y$

$\Rightarrow$ (4) $1 + y^2 + y^2 = 1$

$\Rightarrow 2y^2 = 0 \Rightarrow y = \pm 0$ not possible

No critical points in Case 2.

So, $f(1,0,0) = 1$) max and min

No critical points in Case 2.

So, $\left.\begin{array}{l} f(1,0,0)=1 \\ f(-1,0,0)=-1 \end{array}\right\}$ max and min on D

$\blacksquare$