Problem 1 | Chapter 13.4

Let $u = u(x,y)$ and let (r,θ) be polar coordinates. Show that
$$\|\nabla u\|^2 = u_r^2 + \frac{1}{r^2} u_\theta^2.$$

Thm (chain rule) Let $g: \mathbb{R}^n \to \mathbb{R}^m$ and $f: \mathbb{R}^m \to \mathbb{R}^p$ be differentiable functions such that $f \circ g$ is defined. Then $f \circ g$ is differentiable and $D f\circ g(x_0)$ is a $p \times n$ matrix given by

$$\underbrace{D(f\circ g)(x_0)}_{p \times n} = \underbrace{Df(g(x_0))}_{p \times m} \cdot \underbrace{Dg(x_0)}_{m \times n}$$

Solution Consider the function $h: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h(r,\theta) = (r\cos\theta, r\sin\theta)$.

Apply chain rule to $u \circ h$:

fnc of r,θ fnc of x,y

$$[u_r \; u_\theta] = D(u \circ h) = Du\, Dh$$
$$= [u_x \; u_y] \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$
$$= \begin{bmatrix} u_x \frac{\partial x}{\partial r} + u_y \frac{\partial y}{\partial r} & u_x \frac{\partial x}{\partial \theta} + u_y \frac{\partial y}{\partial \theta} \end{bmatrix}$$

So equating entries yields
$$u_r = u_x \frac{\partial x}{\partial r} + u_y \frac{\partial y}{\partial r} = u_x \cos\theta + u_y \sin\theta \qquad u_\theta = u_x \frac{\partial x}{\partial \theta} + u_y \frac{\partial y}{\partial \theta} = -r u_x \sin\theta + r u_y \cos\theta$$

$$\boxed{\|\nabla u\|^2 = u_r^2 + \frac{1}{r^2} u_\theta^2.}$$

$$u_r^2 + \frac{1}{r^2} u_\theta^2 = (u_x \cos\theta + u_y \sin\theta)^2 + \frac{1}{r^2}(-r u_x \sin\theta + r u_y \cos\theta)^2$$
$$= u_x^2 \cos^2\theta + \cancel{2 u_x u_y \sin\theta\cos\theta} + u_y^2 \sin^2\theta$$
$$+ \frac{1}{\cancel{r^2}}\left(\cancel{r^2} u_x^2 \sin^2\theta - \cancel{2 r^2 u_x u_y \sin\theta\cos\theta} + \cancel{r^2} u_y^2 \cos^2\theta\right)$$
$$= u_x^2(\sin^2\theta + \cos^2\theta) + u_y^2(\sin^2\theta \cos^2\theta)$$
$$= u_x^2 + u_y^2 = \|\nabla u\|^2 \qquad \blacksquare$$

Problem 2

Chapter 13.4

Let $f(x,y) = e^{x+y}$ and make the change of variables $\underline{x} = r^5$, $\underline{y} = rs$. Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial r}$.

Solution By the chain rule:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

we have

$$\frac{\partial f}{\partial s} = e^{x+y}\cdot 0 + e^{x+y} r$$

$$= e^{x+y} r = r e^{r^5+rs}$$

and

$$\frac{\partial f}{\partial r} = e^{x+y} 5r^4 + e^{x+y} s$$

$$= e^{x+y}(s + 5r^4)$$

$$= e^{r^5+rs}(s + 5r^4)$$

■

Problem 3 | Chapter 13.5

Find the rate of change of $f(x,y,z) = xyz$ in the direction normal to the surface defined by the equation $yx^2 + yz^2 + xy^2 = 3$ at $(1,1,1)$.

<u>Solution</u> Recall that if $g(x,y,z) = K$ are the level curves of a function g, then ∇g is normal to the level curves.

So define $g(x,y,z) = yx^2 + yz^2 + xy^2$. Then the surface is a level curve of g, so $\nabla g(1,1,1)$ is normal to the surface at $(1,1,1)$. We compute

$$\nabla g(x,y,z) = (g_x, g_y, g_z)$$
$$= (2xy + y^2, x^2 + z^2 + 2xy, 2yz)$$

so $\nabla g(1,1,1) = (3, 4, 2) \Rightarrow \|\nabla g(1,1,1)\| = \sqrt{9+16+4}$
$= \sqrt{29}$

Normalize $\Rightarrow \dfrac{(3,4,2)}{\sqrt{29}}$

Now we find the directional derivative of f at $(1,1,1)$ in this direction:

$$\nabla f(x) \cdot \frac{(3,4,2)}{\sqrt{29}} = (yz, xz, xy)\frac{(3,4,2)}{\sqrt{29}}$$
$$= (1,1,1) \cdot \frac{(3,4,2)}{\sqrt{29}}$$
$$= \frac{1}{\sqrt{29}}(3+4+2) = \frac{9}{\sqrt{29}} \blacksquare$$

Problem 4

Chapter 13.5

Find the points on the hyperboloid $x^2+4y^2-z^2=4$ where the tangent plane is parallel to the plane $2x+2y+z=5$

Solution Recall: two planes are parallel if and only if their normal vectors are parallel. If n_1 and n_2 are the normal vectors, then there exist $c \in \mathbb{R}$ such that $n_1 = c n_2$.

Let (x_0, y_0, z_0) be a point on the hyperboloid. Define $f(x,y,z) = x^2+4y^2-z^2$. then $\nabla f(x_0,y_0,z_0)$ is normal to the hyperboloid.

$$\nabla f(x,y,z) = (2x, 8y, -2z).$$

The normal vector to the plane $2x+2y+z=5$ is $(2,2,1)$. Suppose there is a number $c \in \mathbb{R}$ such that

$$(2x_0, 8y_0, -2z_0) = c(2,2,1).$$

we have the equations: $\begin{cases} 2x_0 = 2c \\ 8y_0 = 2c \\ -2z_0 = c \end{cases} \Rightarrow \underline{(x_0,y_0,z_0)} = \left(c, \frac{1}{4}c, -\frac{1}{2}c\right)$

we also know $x_0^2 + 4y_0^2 - z_0^2 = 4$

$$\Rightarrow c^2 + 4\left(\tfrac{1}{4}c\right)^2 - \left(-\tfrac{1}{2}c\right)^2 = 4$$

$$\Rightarrow c^2 + \tfrac{1}{4}c^2 - \tfrac{1}{4}c^2 = 4 \Rightarrow c^2 = 4$$

so $c = \pm 2$

Hence, $\boxed{(x_0,y_0,z_0) = \left(2, \tfrac{1}{2}, -1\right) \text{ or } \left(-2, -\tfrac{1}{2}, 1\right)}$ ∎

Problem 5

Chapter 13.4

If $z = f(x-y)$, show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.

Solution

Define $h: \mathbb{R}^2 \longrightarrow \mathbb{R}$ via $h(x,y) = x - y$ so that $z(x,y) = (f \circ h)(x,y)$

Apply chain rule to z:

$$\begin{aligned} \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} = Dz &= Df\, Dh \\ &= [f'] \begin{bmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix} \qquad (f: \mathbb{R} \to \mathbb{R}) \\ &= f' \begin{bmatrix} 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} f' & -f' \end{bmatrix} \end{aligned}$$

Now, $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = f' + (-f') = 0$ ∎