Problem 1

Compute the following limits or prove that they do not exist:

(a) \( \lim_{(x, y) \to (0, 0)} \frac{xy}{x^2 + y^2 + 2} \)

(b) \( \lim_{(x, y) \to (0, 0)} \frac{(x-y)^2}{x^2 + y^2} \)

(c) \( \lim_{(x, y) \to (0, 0)} (3x^2 + 3y^2) \log(x^2 + y^2) \)

(d) \( \lim_{(x, y) \to (0, 0)} \frac{y}{x^2 + y^2} \)

(a) \( \lim_{(x, y) \to (0, 0)} \frac{xy}{x^2 + y^2 + 2} \)

\( xy \) is continuous at \((0,0)\), and \(x^2 + y^2 + 2\) is constant at \((0,0)\) and non-zero, so the function \( f(x, y) = \frac{xy}{x^2 + y^2 + 2} \) is continuous at \((0,0)\). So

\[ \lim_{(x, y) \to (0, 0)} f(x, y) = f(0, 0) = 0 \]

(b) \( \lim_{(x, y) \to (0, 0)} \frac{(x-y)^2}{x^2 + y^2} \)

The function \( f(x, y) = \frac{(x-y)^2}{x^2 + y^2} \) is not continuous at \((0,0)\). Let's try to show that the limit does not exist. Strategy: find two paths in \(xy\)-plane that approach \((0,0)\) but give different values in the limit.

Consider the path \( y = x \) and \( x > 0 \):

\[ \lim_{(x, x) \to (0, 0)} \frac{(x-x)^2}{x^2 + x^2} = \lim_{x \to 0} \frac{0}{2x^2} = 0. \]

But along the path where \( y = 0, x > 0 \) we have

\[ \lim_{x \to 0} \frac{(x-0)^2}{x^2 + 0^2} = \lim_{x \to 0} \frac{x^2}{x^2} = \lim_{x \to 0} 1 = 1. \]

(c) \( \lim_{(x, y) \to (0, 0)} (3x^2 + 3y^2) \log(x^2 + y^2) \)

The function is not continuous at \((0,0)\). Let's convert to polar coordinates:

\[ r = r(x, y) = \sqrt{x^2 + y^2} \Rightarrow r^2 = x^2 + y^2 \]

Note that
\[ r = r(x, y) = \sqrt{x^2 + y^2} \implies r^2 = x^2 + y^2 \quad \text{Note that} \]
\[
\lim_{(x, y) \to (0, 0)} r(x, y) = 0. \text{ So we have}
\]
\[
\lim_{r \to 0} 3r^2 \log r^2 = \lim_{r \to 0} \frac{\log r^2}{r^{3}}
\]
\[
= \lim_{r \to 0} \frac{\frac{1}{r^2} \cdot 2r}{-\frac{2}{3}r^3}
\]
\[
= \lim_{r \to 0} \frac{2}{3r^3} = \lim_{r \to 0} -\frac{2}{3r^3} = 0
\]

(d) \[ \lim_{(x, y) \to (0, 0)} \frac{y}{x^2 + y^2} \]

The function is not continuous at \((0, 0)\). Let's convert to polar:
\[ r = \sqrt{x^2 + y^2} \quad y = r \sin \theta \]

We have
\[
\lim_{r \to 0} \frac{r \sin \theta}{r^2} = \lim_{r \to 0} \frac{\sin \theta}{r}
\]

Fix \( \theta = 0 \), then \( \sin \theta = 0 \) so we get
\[
\lim_{r \to 0} \frac{0}{r} = 0.
\]

Fix \( \theta = \frac{\pi}{2} \), then \( \sin \theta = 1 \) so we get
\[
\lim_{r \to 0} \frac{1}{r} = \begin{cases} +\infty & \text{when } r \to 0^+ \\ -\infty & \text{when } r \to 0^- \end{cases}
\]
Problem 2

Compute the derivative $Dy(x)$ for each function:

(a) $f: \mathbb{R}^2 \to \mathbb{R}^2 \quad f(x,y) = \left( x + e^{x^2+y}, yx^3 \right)$

(b) $f: \mathbb{R}^3 \to \mathbb{R}^3 \quad f(x,y,z) = \left( xe^{y^2}, \cos y, x + e^z \right)$

(c) $f: \mathbb{R}^3 \to \mathbb{R}^3 \quad f(\rho, \theta, \phi) = \left( \rho \sin \rho \cos \theta, \rho \sin \rho \sin \theta, \rho \cos \phi \right)$

---

**Definition:**

If $f: \mathbb{R}^n \to \mathbb{R}^m$ (with component functions $f_i: \mathbb{R}^n \to \mathbb{R}, 1 \leq i \leq m$) is differentiable, then the derivative $Df$ is the matrix whose $(i,j)$ entry is $\frac{\partial f_i}{\partial x_j}$:

$$
Df(x_1, \ldots, x_n) = 
\begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}
\end{bmatrix}
$$


---

(a) $f: \mathbb{R}^3 \to \mathbb{R}^2 \quad f(x,y,z) = \left( x + e^{x^2+y}, yx^3 \right)$

**Solution:**

$f_1(x,y,z) = x + e^{x^2+y}$

$f_2(x,y,z) = yx^3$

Note that $Df$ is the $2 \times 3$ matrix:

$$
Df(x,y,z) = 
\begin{bmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z}
\end{bmatrix} = 
\begin{bmatrix}
1 & 1 & e^z \\
2xy & x^2 & 0
\end{bmatrix}
$$

(b) $f: \mathbb{R}^2 \to \mathbb{R}^3 \quad f(x,y) = \left( xe^{y^2}, \cos y, x + e^z \right)$

$Df$ is the $3 \times 2$ matrix:

$$
Df = 
\begin{bmatrix}
e^{y^2} & xe^{y^2} \\
0 & 1 \\
1 & e^z
\end{bmatrix}
$$

(c) $f: \mathbb{R}^3 \to \mathbb{R}^3 \quad f(p, \theta, \phi) = \left( \rho \sin \rho \cos \theta, \rho \sin \rho \sin \theta, \rho \cos \phi \right)$

$Df$ is the $3 \times 3$ matrix:

---
$D_f$ is the $3 \times 3$ matrix:

$$D_f = \begin{bmatrix}
\sin \phi \cos \theta & -\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\
\sin \phi \sin \theta & \rho \sin \phi \cos \theta & \rho \cos \phi \sin \theta \\
\cos \phi & 0 & -\rho \sin \phi
\end{bmatrix}$$
**Problem 3**

Let \( f(x,y) = x e^{y^2} - y e^{x^2} \). (a) Find an equation of the plane tangent to the graph of \( f \) at \((1,2)\). (b) Which point on the surface \( z + y^2 - x^2 = 0 \) has a tangent plane parallel to the one in (a)?

---

**Definition:** If \( f : \mathbb{R}^2 \to \mathbb{R} \) is differentiable at \((x_0, y_0)\), then the plane tangent to the graph of \( f \) at \((x_0, y_0, f(x_0, y_0))\) is given by

\[
2 - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)
\]

**Note:** The normal vector is

\[
h = (f_x(x_0, y_0), f_y(x_0, y_0), -1) = \nabla (f - 2)(x_0, y_0)
\]

(a) We have

\[
f_x(x, y) = e^{y^2} - 2xy e^{x^2}
\]

and

\[
f_x(1,2) = e^4 - 4e
\]

and

\[
f_y(x, y) = 2xy e^{y^2} - e^{x^2}
\]

and

\[
f_y(1,2) = 4e^4 - e
\]

and

\[
f(1,2) = e^4 - 2e
\]

So the tangent plane is given by

\[
z - (e^4 - 2e) = (e^4 - 4e)(x - 1) + (4e^4 - e)(y - 2)
\]

(b) Which point on the surface \( z + y^2 - x^2 = 0 \) has a tangent plane parallel to the one in (a)?

**Solution:** Note two planes are parallel if their normal vectors are parallel. Also, notice that the surface defined by \( z + y^2 - x^2 = 0 \) is the graph of the function...
by $z + y^2 - x^2 = 0$ is the graph of the function

$$g(x, y) = z = x^2 - y^2$$

By the above, the normal vector at a point $(a, b, g(a, b))$ is $(9x, 9y, -1) = (2a, -2b, -1)$

We need to find $a$ and $b$.

The vector $(2a, -2b, -1)$ is parallel to $(e^y - 4e, 4e^y - e, -1)$ if there is number $c$ such that

$$(2a, -2b, -1) = c(e^y - 4e, 4e^y - e, -1)$$

So $-1 = -1 \cdot c$ $\Rightarrow$ $c = 1$ so we have

$$2a = e^y - 4e \quad \text{and} \quad -2b = 4e^y - e$$

So $a = \frac{e^y - 4e}{2}$ and $b = \frac{e - 4e^y}{2}$.
Problem 41

Compute the limits: (a) \( \lim_{(x,y) \to (0,0)} \frac{e^{xy} - 1}{y} \) (b) \( \lim_{(x,y) \to (0,0)} \frac{\cos (xy) - 1}{x^2 y^2} \)