Problem 1 **Chapter 13.2**

Compute the following limits or prove that they do not exist:

(a) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2+2}$

(b) $\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$

(c) $\lim_{(x,y)\to(0,0)} (3x^2+3y^2)\log(x^2+y^2)$

(d) $\lim_{(x,y)\to(0,0)} \frac{y}{x^2+y^2}$

(a) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2+2}$

xy is continuous at $(0,0)$, and x^2+y^2+2 is cont. at $(0,0)$ and non-zero so the function $f(x,y)=\frac{xy}{x^2+y^2+2}$ is continuous at $(0,0)$. So

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$$

(b) $\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$ The function $f(x,y)=\frac{(x-y)^2}{x^2+y^2}$ is <u>not</u> continuous at $(0,0)$. Let's try to show that the limit does not exist. Strategy: find two paths in xy-plane that approach $(0,0)$ but give different values in the limit.

Consider the path $y=x$ and $x>0$:

$$\lim_{(x,x)\to(0,0)} \frac{(x-x)^2}{x^2+x^2} = \lim \frac{0}{2x^2} = 0.$$

But along the path where $y=0$, $x>0$ we have

$$\lim_{x\to 0} \frac{(x-0)^2}{x^2+0^2} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1$$
$$= 1.$$

(c) $\lim_{(x,y)\to(0,0)} (3x^2+3y^2)\log(x^2+y^2)$

The function is not continuous at $(0,0)$. Let's convert to polar coordinates:

$r = r(x,y) = \sqrt{x^2+y^2} \Rightarrow r^2 = x^2+y^2$ Note that

$r = r(x,y) = \sqrt{x^2+y^2} \Rightarrow r^2 = x^2+y^2$ Note that $\lim_{(x,y)\to(0,0)} r(x,y) = 0$. So we have

$$\lim_{r\to 0} 3r^2 \log r^2 = \lim_{r\to 0} \frac{\log r^2}{1/3r^2}$$

$$\overset{L'H}{=} \lim_{r\to 0} \frac{\frac{1}{r^2}\cdot 2r}{-\frac{2}{3}r^{-3}}$$

$$= \lim_{r\to 0} \frac{\frac{2}{r}}{-\frac{2}{3r^3}} = \lim_{r\to 0} -3r^2$$

$$= 0$$

(d) $\lim_{(x,y)\to(0,0)} \frac{y}{x^2+y^2}$

The function is not continuous at $(0,0)$. Let's convert to polar:

$r = \sqrt{x^2+y^2}$ $\quad y = r\sin\theta$

We have

$$\lim_{r\to 0} \frac{r\sin\theta}{r^2} = \lim_{r\to 0} \frac{\sin\theta}{r}$$

Fix $\theta = 0$, then $\sin\theta = 0$ so we get

$$\lim_{r\to 0} \frac{0}{r} = 0.$$

Fix $\theta = \frac{\pi}{2}$, then $\sin\theta = 1$ so we get

$$\lim_{r\to 0} \frac{1}{r} = \begin{cases} +\infty & \text{when } r\to 0^+ \\ -\infty & \text{when } r\to 0^- \end{cases}$$

Problem 2

Chapter 13.3

Compute the derivative $Df(x)$ for each function:

(a) $f:\mathbb{R}^2 \to \mathbb{R}^2$ $f(x,y) = (x+e^z+y, yx^2)$

(b) $f:\mathbb{R}^2 \to \mathbb{R}^3$ $f(x,y) = (xe^y + \cos y, x, x+e^y)$

(c) $f:\mathbb{R}^3 \to \mathbb{R}^3$ $f(\rho,\theta,\phi) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$

Def If $f:\mathbb{R}^n \to \mathbb{R}^m$ (w/ component functions $f_i:\mathbb{R}^n \to \mathbb{R}$, $1\le i\le m$) is differentiable, then the derivative Df is the $m\times n$ matrix whose ij entry is $\frac{\partial f_i}{\partial x_j}$:

$$Df(x_1,\dots,x_n) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

(a) $f:\mathbb{R}^3 \to \mathbb{R}^2$ $f(x,y) = (x+e^z+y, yx^2)$

Solution: $f_1(x,y,z) = x+e^z+y$

$f_2(x,y,z) = yx^2$

Note that Df is the 2×3 matrix:

$$Df(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & 1 & e^z \\ 2xy & x^2 & 0 \end{bmatrix}$$

(b) $f:\mathbb{R}^2 \to \mathbb{R}^3$ $f(x,y) = (xe^y + \cos y, x, x+e^y)$

Df is the 3×2 matrix:

$$Df = \begin{bmatrix} e^y & xe^y - \sin y \\ 1 & 0 \\ 1 & e^y \end{bmatrix}$$

(c) $f:\mathbb{R}^3 \to \mathbb{R}^3$ $f(\rho,\theta,\phi) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$

Df is the 3×3 matrix:

Df is the 3x3 matrix :

$$Df = \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}$$

Problem 3 | Chapter 13.3

Let $f(x,y) = xe^{y^2} - ye^{x^2}$. (a) Find an eq. of the plane tangent to the graph of f at $(1,2)$. (b) Which point on the surface $z + y^2 - x^2 = 0$ has a tangent plane parallel to the one in (a)?

Def If $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ is differentiable at (x_0, y_0), then the plane tangent to the graph of f at $(x_0, y_0, f(x_0,y_0))$ is given by

$$z - f(x_0,y_0) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Note that the normal vector is

$$n = (f_x(x_0,y_0), f_y(x_0,y_0), -1) = \nabla(f-z)(x_0,y_0)$$

(a) We have
$$f_x(x,y) = e^{y^2} - 2xye^{x^2}$$
$$f_x(1,2) = e^4 - 4e$$

and
$$f_y(x,y) = 2xye^{y^2} - e^{x^2}$$
$$f_y(1,2) = 4e^4 - e$$

and
$$f(1,2) = e^4 - 2e$$

So the tangent plane is given by

$$\boxed{z - (e^4 - 2e) = (e^4 - 4e)(x-1) + (4e^4 - e)(y-2)}$$

(b) Which point on the surface $z + y^2 - x^2 = 0$ has a tangent plane parallel to the one in (a)?

Solution Note two planes are parallel if their normal vectors are parallel. Also, notice that the surface defined by $z + y^2 - x^2 = 0$ is the graph of the function by $z + y^2 - x^2 = 0$ is the graph of the function

$$g(x,y) = z = x^2 - y^2$$

By the above, the normal vector at a point $(a, b, g(a,b))$ is $(g_x, g_y, -1) = (2a, -2b, -1)$

We need to find a and b.

The vector $(2a, -2b, -1)$ is parallel to $(e^4 - 4e, 4e^4 - e, -1)$ if there is number c such that

$$(2a, -2b, -1) = c\,(e^4 - 4e, 4e^4 - e, -1)$$

So $-1 = -1 \cdot c \Rightarrow c = 1$ so we have

$$2a = e^4 - 4e \quad \text{and} \quad -2b = 4e^4 - e$$

So $a = \dfrac{e^4 - 4e}{2}$ and $b = \dfrac{e - 4e^4}{2}$.

Problem 4

Chapter 13.2

Compute the limits: (a) $\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{y}$ (b) $\lim_{(x,y)\to(0,0)} \frac{\cos(xy)-1}{x^2y^2}$