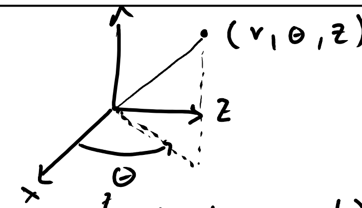
Problem 1 | **Chapter 11.5**

Sketch the following surfaces: (a) $z = r^2$
(b) $\rho = 4\csc\phi\sec\theta$
(c) $r = 4\sin\theta$

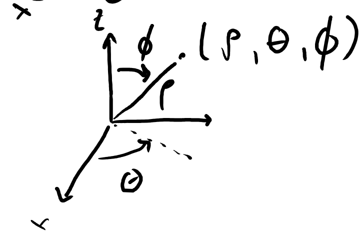
Cylindrical Coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



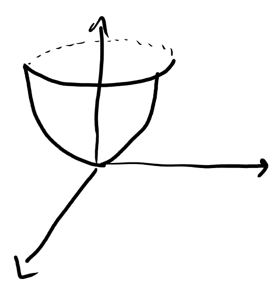
Spherical Coordinates

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$



Solution (a) $z = r^2 = x^2 + y^2$ (since $r = \sqrt{x^2+y^2}$)

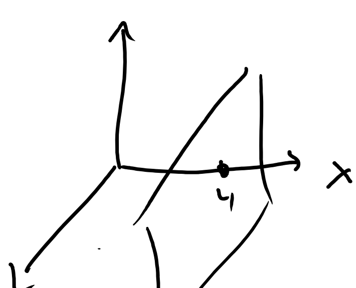
So the surface is defined by $z = x^2+y^2$ which is a paraboloid



(b) $\rho = 4\csc\phi\sec\theta \Rightarrow 4 = \dfrac{\rho}{\csc\phi\sec\theta}$

$$= \rho\sin\phi\cos\theta$$
$$= x$$

So the surface is defined by $x = 4$ which is a plane



(c) $r = 4\sin\theta \Rightarrow r^2 = 16\sin^2\theta$

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$$\Rightarrow x^2+y^2 = 4\cdot(4\sin\theta)\cdot\sin\theta$$
$$= 4r\sin\theta$$
$$= 4y$$

$\Rightarrow x^2+y^2-4y=0$

$\Rightarrow x^2+(y-2)^2 = 4 \Rightarrow$ equation for a sphere of radius 2 centered at $(0,2)$. $\blacksquare$

$\boxed{\text{Problem 2}}$ $\boxed{\text{Chapter 11.7}}$

(a) For any $n \times n$ matrix A, show that $\det(\underline{cA}) = c^n \det(A)$

(b) Compute $\det(M)$ where $M = \begin{bmatrix} 3 & 3 & 0 \\ 0 & 6 & 6 \\ 3 & 0 & -9 \end{bmatrix}$. Is M invertible?

(a) Proof We will use the fact $\det(B \cdot C) = \det(B) \cdot \det(C)$ for any $n \times n$ matrices B and C.

Let $c \in \mathbb{R}$ and $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ be any $n \times n$ matrix. We can write

$$cA = \underbrace{\begin{bmatrix} c & 0 & \cdots & 0 \\ 0 & c & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & c \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}}_{A}$$

By determinant formula

$$\det(cA) = \det(XA)$$
$$= \det(X) \cdot \det(A) = c^n \det A.$$

$\blacksquare$

(b) $M = \begin{bmatrix} 3 & 3 & 0 \\ 0 & 6 & 6 \\ 3 & 0 & -9 \end{bmatrix} = 3 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 1 & 0 & -3 \end{bmatrix}$

$$\det(M) = \det\left(\begin{bmatrix} 3 & 3 & 0 \\ 0 & 6 & 6 \\ 3 & 0 & -9 \end{bmatrix} \right)$$
$$= 3^3 \det\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 1 & 0 & -3 \end{bmatrix} \right) = 27 \left(1 \begin{vmatrix} 2 & 2 \\ 0 & -3 \end{vmatrix} - 1 \begin{vmatrix} 0 & 2 \\ 1 & -3 \end{vmatrix} \right)$$
$$= 27 \left(-6 + 2 \right) = 27 \cdot -4$$
$$= -108.$$

Is M invertible? Thm M is invertible if and only if $\det(M) \neq 0$.

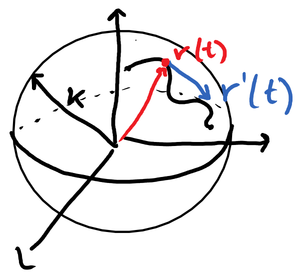
$\blacksquare$

Problem 3

Chapter 12.1

Suppose that $r(t)$ is a space curve with the property that $\|r(t)\| = K$ for all $t \in \mathbb{R}$. Show that $r(t)$ and $r'(t)$ are orthogonal.

Note that $\|r(t)\| = k \iff$ all the points on the curve r lie on a sphere of radius k centered at the origin



Proof We need to show that $r(t) \cdot r'(t) = 0$.

We have

$$\begin{aligned} 0 &= \frac{d}{dt} K^2 \\ &= \frac{d}{dt} \|r(t)\|^2 \\ &= \frac{d}{dt}\left(r(t)\cdot r(t)\right) \\ &= r'(t)\cdot r(t) + r(t)\cdot r'(t) \\ &= 2\, r(t)\cdot r'(t). \end{aligned}$$

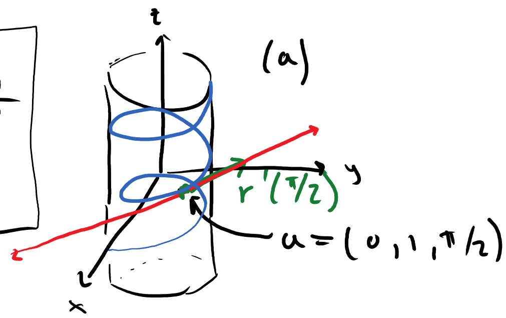
(use product rule for $\cdot$ $(u\cdot v)' = u'\cdot v + v'\cdot u$)

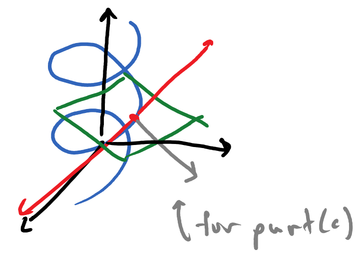
Divide by 2 to get $r(t)\cdot r'(t) = 0$ which proves that $r(t) \perp r'$

Problem 4

Consider the path $r(t) = (\cos t, \sin t, t)$.
(a) Find an equation of the line tangent to r when $t = \pi/2$.
(b) Find a plane orthogonal to the tangent line when $t = \pi/2$.
(c) Find a plane that contains the tangent line when $t = \pi/2$.

$r(t)$ is a helix that lies on a cylinder of radius 1, centered at the origin





(a) A point in the line is given by
$$a = r(\pi/2) = (\cos \pi/2, \sin \pi/2, \pi/2) = (0, 1, \pi/2)$$

A vector pointing in the direction of the line is
$$v = r'(\pi/2) = (-\sin \pi/2, \cos(\pi/2), 1) = (-1, 0, 1)$$

So a parameterization of the tangent line is
$$\ell(t) = (0, 1, \pi/2) + t(-1, 0, 1) = (-t, 1, t + \pi/2).$$

(b) Find a plane orthogonal to the tangent line when $t = \pi/2$.

A point in the plane is $a = r(\pi/2) = (0, 1, \pi/2)$ and a normal vector to the plane is $v = r'(\pi/2) = (-1, 0, 1)$. So an equation of the plane is
$$(-1, 0, 1) \cdot ((x, y, z) - (0, 1, \pi/2)) = 0$$
$$\Rightarrow \boxed{-x + z = \pi/2}$$

(c) Find a plane that contains the tangent line when $t = \pi/2$.
Recall that $v = r'(\pi/2) = (-1, 0, 1)$ is parallel to the tangent line.

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We need is a vector orthogonal to v.

Define a vector $T(\pi/2) = \dfrac{v(\pi/2)}{\|v(\pi/2)\|} = \dfrac{(-1, 0, 1)}{\sqrt{2}} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

Notice that T is parallel to v and $\|T(t)\| = 1$. By Problem (3), we know that $T(t)$ and $T'(t)$ are orthogonal.

Compute $T'(t) = \left(\dfrac{r'(t)}{\|r'(t)\|}\right)' = \left(\dfrac{(-\sin t, \cos t, 1)}{\sqrt{2}}\right)'$

$$= \left(-\frac{1}{\sqrt{2}}\cos t, -\frac{1}{\sqrt{2}}\sin t, 0\right)$$

So a normal vector is $\sqrt{2}\,T'(\pi/2) = \left(0, -\frac{1}{\sqrt{2}}, 0\right) \cdot \sqrt{2}$

$$= (0, -1, 0)$$

So an equation of the plane is

$$(0, -1, 0)\big((x, y, z) - (0, 1, \pi/2)\big) = 0$$