Math 23A, Summer 2019 S  Final EXAM  UCSC, Tromba & Blumberg, 8/30/19

Notes: No calculators are allowed. Please put away your cell phones and other electronic devices, including smart watches, turned off or in airplane mode. Put your answers in the provided boxes.

Warning: No discussing will be tolerated. Students who communicate with other students during the exam must immediately turn in their exams.

Your Name:  
Your Score: /100 Last Initial 

1. (4 points) Find the equation of the tangent line to the graph of \( f(x,y) = x^2 + y^2 - xy \) at the point \((1,1)\).

   (a) \( x - 1 = x^2 - xy + y^2 \)  
   (b) \( y = x^2 - xy + y^2 \)  
   (c) \( x = x^2 - xy + y^2 \)  
   (d) \( y = x^2 - xy + y^2 \)

   Answer (Letter) C

2. (4 points) Find the equation of the tangent plane to the surface \( S \) defined by the equation \( 5x^2 + 3y^2 = 1 \) at the point \((0,0,1)\).

   (a) \( 3x + 6y + 6z = 0 \)  
   (b) \( 3x + 6y - 6z = 0 \)  
   (c) \( 3x + 6y - 6z = 1 \)  
   (d) \( 3x + 6y + 6z = 1 \)

   Answer (Letter) B

3. (10 points) For each of the questions below, indicate if the statement is true (T) or false (F).
   (a) If \( v_1 \) and \( v_2 \) are orthogonal nonzero vectors in \( \mathbb{R}^3 \), then \( v_1 \cdot v_2 = 0 \)

   Answer (T/F) T

   (b) Let \( v \) be a non-zero constant. The function \( f(x,y) = \cos(x) \sin(y) \) satisfies the differential equation \( \frac{\partial f}{\partial x} + \frac{1}{x} \frac{\partial f}{\partial y} = \frac{1}{x} \frac{\partial f}{\partial y} \)

   Answer (T/F) T

   (c) Let \( f: \mathbb{R}^2 \to \mathbb{R} \) be a \( C^2 \) function of two variables \( x \) and \( y \). Then \( \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \)

   Answer (T/F) T

   (d) \( \nabla \times \mathbf{F} = 0 \) for \( \mathbf{F} \in \mathbb{R}^3 \) be a \( C^2 \) vector field. Then div(\( \nabla \times \mathbf{F} \)) = div(cross(F)) = 0.

   Answer (T/F) T

   (e) The line \( \{(x,y,z) = (2 + t, 3 + t, t - 2) \mid t \in \mathbb{R} \} \) is in the plane \( \{z = x - y\} \).

   Answer (T/F) F

4. (15 points) Write \( y = 5 \sin x + 5 \sin 2x + 5 \sin 5x \), so \( \sqrt{g}(0.1,1) \) is the normal vector:\n
   \( \sqrt{g}(0.1,1) = (5 \cos x, 5 \cos y, 5 \cos 5x) \)

   \( \rightarrow \) scale \( \frac{\hat{n}}{5} \Rightarrow \sqrt{g}(0.1,1) = (1, 1, 1) \).

   Then \( x = y = -1 \) is \( 11, 11 \parallel 0 \) and \( y = 0 \) is \( 11, 11 \parallel 0 \).

   \( \Rightarrow \) the angle between \( v_1, v_2 \) is \( \pi/2 \) or \( 3\pi/2 \). Then \( \cos \theta \neq 0 \) and this is not zero since \( \sin \pi/2 = 1 \) and \( \sin 3\pi/2 = 1 \).

   (f) \( \frac{\partial}{\partial x} (\cos x) = \frac{1}{\cos x} \frac{\partial}{\partial y} (-\cos x \cdot \sin y)\)

   \( = -\cos x \cos y = \frac{d^2}{dx^2} \Rightarrow \text{True} \)

   (g) True, Clairault's Theorem.

   (h) True, proved on weblab (see notes).

   (i) False. Take \( t = 2 \Rightarrow (2) = (4, 8, 0) \). But \( 0 \neq 1 - 8 \). So this point does not lie in the plane.
4. (5 points) Let \( F(x, y, z) = (-x, y, z) \) be a vector field on \( \mathbb{R}^3 \). Which of the following is a flow line \( c(t) \) for \( F^t \)?

(a) \( c(t) = (e^t, e^{-t}, 0) \)
(b) \( c(t) = (-e^t, e^{-t}, 1) \)
(c) \( c(t) = (\sin t, \cos t, e^t) \)
(d) \( c(t) = (\cos t, e^t, -e^{-t}) \)

Answer (Letters) (A) (B) (C) (D) (E) None. (F) All.

5. (8 points) Consider the function

\[
f(x, y) = \begin{cases} \sqrt{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}
\]

(a) Compute \( \frac{\partial f}{\partial x}(0, 0) \)
(b) Compute \( \frac{\partial f}{\partial y}(0, 0) \)

For (c) and (d), state whether the statement is true (T) or false (F).

(c) The function \( f(x, y) \) is continuous at \((0,0)\). Answer (T/F) T

(d) The function \( f(x, y) \) is differentiable at \((0,0)\). Answer (T/F) F

6. (6 points) Find the distance \( d \) from the point \((0, 2)\) in the \( x - y \) plane to the curve \( x^2 - y^2 = 2 \).

\[
d = 2
\]

7. (10 points) Consider \( f(x, y) = x - y^3 + xy^2 \)

(a) (2 points each) Write down the two equations the critical points must satisfy.

\[
1 - 3y^2x + 2y = 0
\]

(b) (2 points) Solve the equations to obtain the two critical points

(i) \((1,0)\) (ii) \((-1,0)\)

(c) (1 point each) State whether (i), (ii) are a local maximum (MAX), local minimum (MIN) or saddle point (SADD).

(i) SADD (ii) SADD

8. (5 points) For \( f(x, y) = x^2 - y^2 \)

(i) \( \frac{\partial}{\partial x} f(x, y) = 2x \)

(ii) \( \frac{\partial}{\partial y} f(x, y) = -2y \)

(iii) Critical points are \((\pm 1, 0)\). \( D(x, y) = 4x^2 + 4y^2 = 2 \)

9. (5 points) Consider \( f(x, y) = x^2 - y^2 \)

(i) \( \frac{\partial}{\partial x} f(x, y) = 2x \)

(ii) \( \frac{\partial}{\partial y} f(x, y) = -2y \)

(iii) Critical points are \((\pm 1, 0)\). \( D(x, y) = 4x^2 + 4y^2 = 2 \)

(iv) \( D < 0 \Rightarrow \) saddle point

\[
D = \begin{vmatrix} 2x & 2y \\ 2x & 2y \end{vmatrix} = 4x^2 - 4y^2 
\]

\[\Rightarrow\text{Saddle Points}!\]
8. (10 points) Use the method of Lagrange Multipliers to find the maximum and minimum values of \( f(x, y) = y^2 \) subject to the constraints \( x^2 + y^2 = 1 \).

(a) Give the Lagrange multiplier equations:

Equation 1: \[ y = 2\lambda x \]  
Equation 2: \[ x = 2\lambda y \]

(b) Give a comma-separated list of all critical points of \( f \) subject to the given constraint:

\[ \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right), \left( -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \]

(c) Give the global maximum and minimum values of \( f \) subject to the given constraint:

Global maximum value of \( f \) is \[ \frac{1}{2} \]  
Global minimum value of \( f \) is \[ -\frac{1}{2} \]

9. (6 points) Let \( f(x, y) = x^n + y^n \) and \( g(x, y) = x^2 + y^2 \). Find \( Df(x, y), Dg(x, y) \) and use the chain rule to compute the derivative \( D(f \circ g)(1,1) \).

- \( Df(x, y) = \begin{pmatrix} n x^{n-1} & n y^{n-1} \end{pmatrix} \)
- \( Dg(x, y) = \begin{pmatrix} 2x & 2y \end{pmatrix} \)
- \( D(f \circ g)(1,1) = Df(1,1) Dg(1,1) \)

10. (8 points) Let \( f(x, y) = \frac{x^2}{x^2 + 1} \). Compute the following:

(a) \( \nabla f(x, y) = \begin{pmatrix} \frac{2x}{(x^2 + 1)^2} & \frac{-2y}{(x^2 + 1)^2} \end{pmatrix} \)

(b) The direction of the fastest rate of increase of \( f \) at the point \( P(1,1) \). Give your answer as a unit vector.

\[ \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \]

(c) The directional derivative (i.e. rate of change) of \( f \) at the point \( P(1,1) \) in the direction \( v = (0,2) \).

\[ \frac{2}{x^2 + 1} \]

(d) Give a direction \( u = (a, b) \) so that the directional derivative of \( f \) at the point \( P(1,1) \) in the direction \( u \) has value 0. Give your answer as a unit vector.

\[ u = \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \]
11. (9 points) A vector field \( \mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3 \) is defined by \( \mathbf{F}(x, y, z) = (-x^2, y^2, 2z(x-y)) \). Compute the following:

(a) (3 points) \( \text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = -2x + y + 2z(x-y) \).

(b) (3 points) \( \text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \left[ \begin{array}{c} -\frac{\partial}{\partial y} \cdot 2z(x-y) \\ \frac{\partial}{\partial z} \cdot (-x^2) \\ \frac{\partial}{\partial x} \cdot y^2 \end{array} \right] = (-2z - 2z, -x^2, y^2) \).

(c) (3 points) True or false: Is \( \mathbf{F} \) a gradient vector field?

12. (6 points)

(a) Find the first order Taylor polynomial (or Taylor approximation) of \( f(x,y) = \sin xy \) at \((0,0)\).

\( f(0,0) = 0 \).

(b) Find the second order Taylor polynomial (or Taylor approximation) of \( f(x,y) = \sin xy \) at \((0,0)\).

\( f(0,0) = 0 \).

13. (6 points) Let \( P \) be the tangent plane to the graph of \( f(x,y) = x^2 - 2y^2 + 3z^2 \) at the point \((1,1,4)\). Let \( f(x,y) = x^2 - 2y^2 + 3z^2 \). Find the point \( Q(x_0,y_0,z_0) \) on the graph of \( f \) where the tangent plane is parallel to \( P \).

\( Q(2,1,2) \).

14. (8 points) Let the particle \( r(t) = (\cos^2 t, \sin^2 t, t) \).

(a) Velocity at \( t = 0 \):

\( (-2 \sin t, 2 \cos t, 1) \).

(b) Speed at \( t = 0 \):

\( \sqrt{20} \).

(c) Acceleration at \( t = 0 \):

\( (11, 0, 1) \).

(d) Equation of the tangent line to \( r(t) \) at \( t = 0 \):

\( (4, 2, 0) + s(1, 0, 1) \).

15. (12 points) \( \nabla \cdot \mathbf{F} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left( \begin{array}{c} f_x \\ f_y \\ f_z \end{array} \right) = 0 \).

(b) \( \nabla \times \mathbf{F} = \left( \begin{array}{c} f_{yx} - f_{yz} \\ f_{zx} - f_{xz} \\ f_{xy} - f_{yx} \end{array} \right) = 0 \).

(c) If \( \mathbf{F} = \nabla f \), then \( \mathbf{F} \times \mathbf{F} = \nabla \times (\mathbf{F} \cdot \mathbf{F}) = 0 \) by then.

16. (12 points)

(a) \( f(\alpha, \beta) = 0 \) \( \Rightarrow \) \( T_i(\alpha, \beta) = \left( \frac{\partial f}{\partial \alpha} \right) \mathbf{e}_\alpha + \left( \frac{\partial f}{\partial \beta} \right) \mathbf{e}_\beta \).

(b) \( T_1(\alpha, \beta) = 0 \).

(c) For the motivated student:

\( T_2(\alpha, \beta) = \left( \frac{\partial f}{\partial \alpha} \right) \mathbf{e}_\alpha + \left( \frac{\partial f}{\partial \beta} \right) \mathbf{e}_\beta \).

17. (13 points) The normal vector for \( P \) is \( \mathbf{n}_1 = (3, 2, 1) \).

Now let \( Q(\alpha, \beta) \) lie on \( f \). The tangent plane to \( f \) at \( Q \) has normal vector \( \mathbf{n}_2 = (f_x(\alpha, \beta), f_y(\alpha, \beta), -1) \).

\( \mathbf{n}_1 = \mathbf{n}_2 \), then \( k = 1 \) so that \( -k + 2 \alpha \) and \( -k = 2 \alpha \) so \( \alpha = 1 \) and \( \beta = 3 \). Then \( \mathbf{c} = \mathbf{n}_1(\alpha, \beta) = (2, 1, 2) \).

18. (11 points)

(a) \( \mathbf{c}'(t) = (4e^{3t}, 2 \cos 2t, -6 \sin t) \) \( \left( 5 \right) \).

(b) \( \mathbf{c}(t) = \left( 5 \right) \).

(c) \( \mathbf{c}'(t) = (15 \cos 2t, -3 \sin 2t, -6 \sin t) \) \( \to \mathbf{v} \).

(d) \( \mathbf{v}(t) = \mathbf{c}'(t) + \mathbf{c}'(t) \mathbf{e} = (1, 0, 1) + (4, 1, 2) \).

(e) \( \mathbf{c}(t) = \mathbf{c}(t) + \mathbf{c}(t) \mathbf{e} = (1, 0, 1) + (1, 2, 1) \).