Math 23A, Summer 2019 S2 **Final EXAM** UCSC, Tromba & Bäuerle, 8/30/19

Notes: No calculators are allowed. **Please put away your cell phones and other electronic devices, including smart watches, turned off or in airplane mode.** Put your answers in the provided boxes.

Warning: No cheating will be tolerated. **Students who communicate with other students during the exam must immediately turn in their exam.**

Your Name: Jadyn **Your Score:** ☐/100 **Last Initial:** ☐

1. (4 points) Find the equation of the tangent plane to the graph of $f(x, y) = 1 + x^2y - xy^2$ at the point $(1, 1)$.
(A) $z = x - y$ (B) $0 = 1 + (x - 1) - (y - 1)$ (C) $z = 1 + (x - 1) - (y - 1)$
(D) $z = 1 - (y^2 - 2xy)(x - 1) + (x^2 - 2xy)(y - 1)$ (E) $z = -1 + (x - 1) - (y - 1)$

Answer (Letter): C

2. (4 points) Find the equation of the tangent plane to the surface S defined by the equation $5 \sin x + 5 \sin y + z^5 = 1$ at the point $(0, 0, 1)$.
(A) $5x + 5y + 5z = 1$ (B) $x + y + z = 1$ (C) $5x + 5y - 5z = 1$
(D) $x + y + 5z = 0$ (E) $x + y + z = 5$

Answer (Letter): B

3. (10 points) For each of the questions below, indicate if the statement is true (T) or false (F).
(a) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal non-zero vectors in $\mathbb{R}^3$, then $\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$

Answer (T/F): T

(b) Let c be a non-zero constant. the function $f(x, y) = \cos(x)\cos(ct)$ satisfies the differential equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

.

Answer (T/F): T

(c) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a C^2 function of two variables x and y. Then

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = \frac{\partial^2 f}{\partial x \partial y}(x, y)$$

.

Answer (T/F): T

(d) Let $\mathbf{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a C^2 vector field. Then $\operatorname{div}(\nabla \times \mathbf{F}) = \operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0$.

Answer (T/F): T

(e) The line $l(t) = (x, y, z) = (2 + t, 2t + 4, t - 2)$ lies in the plane $z = x - y$.

Answer (T/F): F

1

(1) $z - f(1,1) = f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$\Rightarrow z - 1 = 1\cdot(x-1) + (-1)(y-1)$

$\Rightarrow z = 1 + (x-1) - (y-1)$

$f_x = 2xy - y^2$

$f_y = x^2 - 2xy$

(2) Write $g(x,y,z) = 5\sin x + 5\sin y + z^5$, so $\nabla g(0,0,1)$ is the normal vector:

$$\nabla g(0,0,1) = (5\cos x, 5\cos y, 5z^4)\big|_{(0,0,1)} = (5,5,5)$$

$\Rightarrow$ scale $\vec{n} = \frac{1}{5}\nabla g(0,0,1) = (1,1,1)$. The eq is:

$$x + y + z - 1 = (1,1,1)\cdot(x-0, y-0, z-1) = 0$$

(3) (a) True. If $v_1, v_2 \neq 0 \Rightarrow \|v_1\| \neq 0, \|v_2\| \neq 0$. If $v_1 \perp v_2$, $\Rightarrow$ the angle between v_1 & v_2 is $\pi/2$ or $3\pi/2$. Then

$$\|v_1 \times v_2\| = \|v_1\|\|v_2\|\sin(\theta) \neq 0$$

and this is not zero since $\sin \pi/2 = 1$ and $\sin 3\pi/2 = -1$.

(b) $\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \frac{1}{c^2}\frac{\partial}{\partial t}\left(-c\cos(x)\cdot\sin ct\right) = \frac{1}{c^2}\left(-c^2\cos x\cos ct\right)$

$= -\cos x \cos ct = \frac{\partial f^2}{\partial x^2}$. $\Rightarrow$ True

(c) True, Clairaut's Theorem.

(d) True, proved on Wed (see notes).

(e) False. Take $t = 2 \Rightarrow l(2) = (4, 8, 0)$. But $0 \neq 4 - 8$. So this point does not lie in the plane

4. (5 points) Let $\mathbf{F}(x,y,z) = (-x, y, 0)$ be a vector field on $\mathbb{R}^3$. Which of the following is a flow line $\mathbf{c}(t)$ for $\mathbf{F}$?

(a) $\mathbf{c}(t) = (e^t, e^{-t}, 2)$
(b) $\mathbf{c}(t) = (e^{-t}, e^t, 1)$
(c) $\mathbf{c}(t) = (\sin t, e^t, e^{-t})$
(d) $\mathbf{c}(t) = (\sin t, 1, e^t)$

(A) (a) (B) (b) (C) (c) (D) (d) (E) None. (F) All.

Answer (Letter): $\boxed{B}$

5. (8 points) Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \tag{1}$$

(a) Compute $\dfrac{\partial f}{\partial x}(0,0)$ $\qquad \dfrac{\partial f}{\partial x}(0,0) = \boxed{0}$

(b) Compute $\dfrac{\partial f}{\partial y}(0,0)$ $\qquad \dfrac{\partial f}{\partial y}(0,0) = \boxed{0}$

For (c) and (d), state whether the statement is true (T) or false (F)

(c) The function $f(x,y)$ is continuous at $(0,0)$. Answer (T/F): $\boxed{}$

(d) The function $f(x,y)$ is differentiable at $(0,0)$. Answer (T/F): $\boxed{}$

6. (6 points) Find the distance d from the point $(0,2)$ in the $x-y$ plane to the curve $x^2 - y^2 = 2$.

shortest

$d = \boxed{2}$

7. (10 points) Consider $f(x,y) = x - \frac{1}{3}x^3 + xy^2$

(a) (2 points each) Write down the two equations the critical points must satisfy.

$\boxed{1 - x^2 + y^2 = 0}$ $\qquad \boxed{2xy = 0}$

(b) (2 ~~points each) Solve the e~~quations to obtain the two critical points

(i) $\boxed{(1,0)}$ (ii) $\boxed{(-1,0)}$

(c) (1 point each) State whether (i), (ii) are a local maximimum (MAX), local minimum (MIN) or saddle point (SAD).

(i) $\boxed{\text{SAD}}$ (ii) $\boxed{\text{SAD}}$

2

(4) A flow line satisfies $c'(t) = F(c(t))$.

(a) $c'(t) = (e^t, -e^{-t}, 0)$, $F(c(t)) = (-e^t, e^{-t}, 0)$ $\Rightarrow$ not a flow line

(b) $c'(t) = (-e^{-t}, e^t, 0)$ $\rightsquigarrow$ this is a flow line.

(c) $c'(t) = (\cos t, e^t, -e^{-t})$
(d) $c'(t) = (\cos t, 0, e^t)$
$\Big\}$ $\Rightarrow$ not flow lines since z component is not zero. ∎

(5) (a) $\dfrac{\partial f}{\partial x}(0,0) = \lim\limits_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} 0 = 0$

(b) $\dfrac{\partial f}{\partial y}(0,0) = 0$

(c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2} = \lim\limits_{r\to 0} \dfrac{(r\cos\theta)^2(r\sin\theta)^2}{r^2} = \lim\limits_{r\to 0} r^2\sin^2\theta\cos^2\theta$
$= 0 = f(0,0)$.

f is continuous.

(d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y) - f(0,0) - \left[\frac{\partial f}{\partial x}(0,0)\ \frac{\partial f}{\partial y}(0,0)\right]\begin{bmatrix} x-0 \\ y-0 \end{bmatrix}}{\sqrt{x^2+y^2}} = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{(x^2+y^2)^{3/2}}$
$= 0$

So $f(x,y)$ is differentiable at 0.

(6) Lagrange to minimize $d^2(x,y) = x^2 + (y-2)^2$ w/ constraint $x^2 - y^2 = 2$.

$(2x, 2y-4) = \lambda(2x, -2y) \Rightarrow \begin{cases} x = \lambda x \\ y - 2 = -\lambda y \end{cases}$

Case 1 $x = 0 \Rightarrow -y^2 = 2$, nonsense! So $x \neq 0$.

Case 2 $x \neq 0 \Rightarrow \lambda = 1 \Rightarrow y - 2 = -y \Rightarrow y = 1. \Rightarrow x = \pm\sqrt{3}$

So, critical points are $(\pm\sqrt{3}, 1)$. $d(\sqrt{3}, 1) = \sqrt{\sqrt{3}^2 + (1-2)^2} = 2$ ∎

(7) (b) $xy = 0 \Rightarrow x = 0$ or $y = 0$

Case 1 $x = 0 \Rightarrow y^2 = -1 \Rightarrow$ no real solutions

Case 2 $y = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$. So two crit. pts, $(\pm 1, 0)$.

(c) $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2x & 2y \\ 2y & 2x \end{vmatrix} = -4x^2 - 4y^2 \Rightarrow$ @ $(1,0) \rightarrow D < 0$, @ $(-1,0) \Rightarrow D < 0$

$\Rightarrow$ Saddle Points! ∎

8. (10 points) Use the method of Lagrange Multipliers to find the maximum and minimum values of $f(x, y) = xy$ subject to the constraint $4x^2 + y^2 = 1$.

(a) Give the two Lagrange multiplier equations:

Equation 1: $y = 8\lambda x$ Equation 2. $x = 2\lambda y$

(b) Give a comma-separated list of all critical points of f subject to the given constraint:

$\left(\frac{1}{2\sqrt{2}}, \frac{\sqrt{2}}{2}\right), \left(-\frac{1}{2\sqrt{2}}, \frac{\sqrt{2}}{2}\right), \left(\frac{1}{2\sqrt{2}}, -\frac{\sqrt{2}}{2}\right), \left(-\frac{1}{2\sqrt{2}}, -\frac{\sqrt{2}}{2}\right)$

(b) Give the global extreme values of f subject to the given constraint.

Maximum value of f = $\frac{1}{4}$ Minimum value of f = $-\frac{1}{4}$

$f: \mathbb{R}^2 \to \mathbb{R}^3$ $g: \mathbb{R}^2 \to \mathbb{R}^2$

9. (6 points) Let $f(u, v) = (u + v, u - v, uv)$ and $g(x, y) = (x^2 - y^2, x^2 + y^2)$. Find $\mathbf{D}f(u, v)$, $\mathbf{D}g(x, y)$ and use the chain rule to compute the derivative $\mathbf{D}(f \circ g)(0, 1)$.

$\mathbf{D}f(u, v) =$ ✓ $\mathbf{D}g(x, y) =$ ✓ $\mathbf{D}(f \circ g)(0, 1) =$

10. (8 points) Let $f(x, y) = \frac{y^2}{x^4 + 1}$. Compute the following: $y^2(x^4+1)^{-1}$

(a) $\nabla f(x, y)$

$\nabla f(x, y) =$

(b) The direction of the fastest rate of increase of f at the point $P(1, 1)$. Give your answer as a unit vector.

$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(c) The directional derivative (i.e. rate of change) of f at the point $P(1, 1)$ in the direction $\mathbf{v} = (0, 2)$.

1

(d) Give a direction $\mathbf{u} = (u_1, u_2)$ so that the directional derivative of f at the point $P(1, 1)$ in the direction $\mathbf{u}$ has value 0. Give your answer as a unit vector.

$\mathbf{u} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

3

(8) (a) $(y, x) = \lambda(8x, 2y) \Rightarrow y = 8\lambda x, \quad x = 2\lambda y$

(b) $\lambda = 0, \Leftrightarrow y = 0, x = 0$. But $(0,0)$ does not lie on the constraint.

So we have $\frac{y}{x} = \frac{8\lambda x}{2\lambda y} = \frac{4x}{y} \Leftrightarrow y^2 = 4x^2$

$\Rightarrow 1 = 4x^2 + y^2 = 2y^2 \Rightarrow y = \pm \frac{\sqrt{2}}{2}$

$\Rightarrow 1 = 4x^2 + y^2 = 8x^2 \Rightarrow x = \pm\sqrt{\frac{1}{8}} = \pm\frac{1}{2\sqrt{2}}$

(c) $f\left(\frac{1}{2\sqrt{2}}, \frac{\sqrt{2}}{2}\right) = \frac{1}{4} = \max$

$f\left(-\frac{1}{2\sqrt{2}}, \frac{\sqrt{2}}{2}\right) = -\frac{1}{4}$

(9) $Df(u,v) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ v & u \end{bmatrix}$ $D(f \circ g)(0,1) = Df(g(0,1)) \cdot Dg(0,1) = Df(-1,1)\, Dg(0,1)$

$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -4 \\ 0 & -4 \end{bmatrix}$

$Dg(x,y) = \begin{bmatrix} 2x & -2y \\ 2x & 2y \end{bmatrix}$

(10) (a) $\nabla f(x,y) = \left(-y^2(x^4+1)^{-2} \cdot 4x^3, \frac{2y}{x^4+1}\right)$

(b) $\nabla f(1,1) = (-1, 1) \Rightarrow \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(c) $\nabla f(1,1) \cdot \frac{V}{\|V\|} = (-1, 1) \cdot (0, 1) = 1$.

$\frac{V}{\|V\|} = \frac{(0,2)}{\sqrt{0^2+2^2}} = \frac{(0,2)}{2} = (0,1)$

(d) Let $u = (x, y)$. If u is a unit vector, then $x^2 + y^2 = 1$.

Also, $\nabla f(1,1) \cdot u = 0 \Leftrightarrow -x + y = 0 \Rightarrow y = x$

$\Rightarrow y = \pm\frac{\sqrt{2}}{2} \Rightarrow x = \pm\frac{\sqrt{2}}{2}$. So take $u = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

11. (9 points) A vector field $\mathbf{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is defined by $\mathbf{F}(x, y, z) = (-x^2, y^2, 2z(x-y))$. Compute the following:

(a) (3 points) $\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} =$ $\boxed{0}$

(b) (3 points) $\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F} =$ $\boxed{(-2z, -2z, 0)}$

(c) (2 points) True or false: Is $\mathbf{F}$ a gradient vector field ? $\boxed{F}$

12. (6 points)

(a) Find the first order Taylor polynomial (or Taylor approximation) of $f(x, y) = xy\cos(xy)$ at $(0, 0)$.

$T_1(x, y) =$ $\boxed{0}$

(b) Find the second order Taylor polynomial (or Taylor approximation) of $f(x, y) = xy\cos(xy)$ at $(0, 0)$.

$T_2(x, y) =$

13. (6 points) Let $\mathcal{P}$ be the tangent plane to the graph of $g(x, y) = 8 - 2x^2 - 2y^2$ at the point $(1, 1, 4)$. Let $f(x, y) = 6 - x^2 - y^2$. Find the point $Q(x_0, y_0, z_0)$ on the graph of f where the tangent plane is parallel to $\mathcal{P}$.

$Q(x_0, y_0, z_0) =$ $\boxed{(2,2,-2)}$

14. (8 points) Let the path $\mathbf{c}(t) = (e^{4t}, \sin(2t), \cos t)$. Find the velocity, speed, acceleration and the equation of the tangent line to $\mathbf{c}(t)$ at $t = 0$.

(a) Velocity at $t = 0$: $\boxed{(4,2,0)}$

(b) Speed at $t = 0$: $\boxed{\sqrt{20}}$

(c) Acceleration at $t = 0$: $\boxed{(16,0,-1)}$

(d) Equation of the tangent line to $\mathbf{c}(t)$ at $t = 0$: $\boxed{(1+4t, 2t, 1)}$

4

(11) (a) $\nabla \cdot F = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \cdot (-x^2, y^2, 2z(x-y))$

$= -2x + 2y + 2(x-y) = 0.$

(b) $\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -x^2 & y^2 & 2z(x-y) \end{vmatrix} = i(-2z) - j(2z) + k(0-0) = (-2z, -2z, 0)$

(c) If $F = \nabla f$, then $\nabla \times F = \nabla \times (\nabla f) = 0$ by Thm from Wed, see notes.

(12) (a) $f(0,0) = 0$ So $T_1(a,b) = f(0,0) + f_x(0,0)a + f_y(0,0)b$

$f_x(0,0) = 0$ $= 0.$

$f_y(0,0) = 0$

(b) For the motivated student.

$$T_2(a,b) = \underbrace{f(0,0) + f_x(0,0)a + f_y(0,0)b}_{T_1} + \frac{1}{2}\big(f_{xx}(0,0)a^2 + f_{xy}(0,0)ab + f_{yx}(0,0)ba + f_{yy}(0,0)b^2 \big)$$

(13) The normal vector for P is $n_1 = (g_x(1,1), g_y(1,1), -1)$

$= (-4, -4, -1)$

Now let $Q(a,b,c)$ lie on f. The tangent plane to f at Q has normal vector $n_2 = (f_x(a,b), f_y(a,b), -1) = (-2a, -2b, -1)$

If $n_1 = \lambda n_2$, then $\lambda = 1$ so that $-4 = -2a$ and $-4 = -2b$ so $(a,b,c) = (2,2,c)$. Then $c = f(a,b) = -2$.

(14) (a) $c'(0) = (4e^{4t}, 2\cos 2t, -\sin t)\big|_{t=0} = (4, 2, 0)$

(b) magnitude of $c'(0)$

(c) $c''(0) = (16e^{4t}, -4\sin 2t, -\cos t)\big|_{t=0} = (16, 0, -1)$

(d) $l(t) = c(0) + c'(0)t = (1,0,1) + (4,2,0)t$

$= (1+4t, 2t, 1)$