Problem 1 | Chapter 14.1

Show that $f(x,t) = \sin(x-ct)$ satisfies the one-dimensional wave equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial^2 t}$$

Solution

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\sin(x-ct)\right) \\
&= \frac{\partial}{\partial x}\cos(x-ct) \\
&= -\sin(x-ct) \\
&= -\frac{c^2}{c^2}\sin(x-ct) \\
&= -\frac{c}{c^2}\left(c\sin(x-ct)\right) \\
&= -\frac{c}{c^2}\left(\frac{\partial}{\partial t}\cos(x-ct)\right) \\
&= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\sin(x-ct) = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} \quad \blacksquare
\end{aligned}$$

Problem 2 | **Chapter 14.1**

Let $w = f(x,y)$ be a real valued function of 2 variables and let $x = u+v$ and $y = u-v$. Show that $\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}$.

Solution

Lemma: Let $g(x,y)$ be any function of two variables and make the change of variables $x = u+v$ and $y = u-v$. By chain rule:

$$(1)\quad \frac{\partial g}{\partial u} = \frac{\partial g}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial g}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}$$

$$(2)\quad \frac{\partial g}{\partial v} = \frac{\partial g}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial g}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial g}{\partial x} - \frac{\partial g}{\partial y}.$$

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}\right) \quad \left(\text{by eq. (2)}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x}\right) - \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial y}\right)$$

$$\left(\text{by eq (1) applied to } \frac{\partial w}{\partial x} \text{ and } \frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) - \left(\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)\right)$$

$$= \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y \partial x} - \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2}$$

$$\left(\text{by Clairaut's Thm } f_{xy} = f_{yx}\right) = \frac{\partial^2 w}{\partial x^2} + \cancel{\frac{\partial^2 w}{\partial y \partial x}} - \cancel{\frac{\partial^2 w}{\partial y \partial x}} - \frac{\partial^2 w}{\partial y^2}$$

$$= \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \qquad \blacksquare$$

Problem 3

Chapter 14.1

Does there exist a C^2 function $f(x,y)$ such that:

(a) $f_x = 2x-5y$ and $f_y = 4x+y$

(b) $f_x = 5x-2y$ and $f_y = -2x$

Solution

Clairaut's Thm If $f(x,y)$ is C^2 (twice continuously differentiable), then $f_{xy} = f_{yx}$.

(a) There is no such function. Suppose there was a C^2 fnc $f(x,y)$ such that $f_x = 2x-5y$ and $f_y = 4x+y$. By Clairaut's Thm, we know that $f_{xy} = f_{yx}$. But

$$f_{xy} = -5 \quad \text{and} \quad f_{yx} = 4 \quad \text{so} \quad f_{xy} \neq f_{yx}.$$

So no such function exists.

(b) Try the same method as (a): $f_{xy} = -2$ and $f_{yx} = -2$. The mixed partials f_{xy} and f_{yx}, so this tells us nothing.

Let's try to find a function $f(x,y)$ that works:

$f_x = 5x-2y$ and $f_y = -2x$

Given $f_x = 5x-2y \Rightarrow f = \int f_x \, dx = \frac{5}{2}x^2 - 2xy + h(y)$

where $h(y)$ is any function of y.

Then we have

$$-2x = f_y = \frac{\partial}{\partial y}\left(\frac{5}{2}x^2 - 2xy + h(y)\right)$$

$$= -2x + h'(y)$$

$$\Rightarrow h'(y) = 0 \Rightarrow h(y) = \int h'(y)\, dy = \int 0 \, dy = 0$$

So $f(x,y) = \frac{5}{2}x^2 - 2xy$ is C^2 function satisfying the given properties. ∎

Problem 4 | Chapter 14.2

Find the first and second-order Taylor approximation for $f(x,y) = 6e^{x+y}$ at $(0,0)$.

Theorem 3 Second-Order Taylor Formula Let $f: U \subset \mathbb{R}^n \to \mathbb{R}$ have continuous partial derivatives of third order.[2] Then we may write

$$f(\mathbf{x}_0+\mathbf{h}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{x}_0) + \frac{1}{2}\sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) + R_2(\mathbf{x}_0, \mathbf{h}),$$

where $R_2(\mathbf{x}_0, \mathbf{h})/\|\mathbf{h}\|^2 \to 0$ as $\mathbf{h} \to \mathbf{0}$ and the second sum is over all i's and j's between 1 and n (so there are n^2 terms).

The theorem says that f can be approximated in a neighborhood of x_0 by the polynomial

$$T(h_1, \dots, h_n) = \underbrace{f(x_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(x_0)}_{\text{first order eq.}} + \frac{1}{2}\sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)$$

In our case, we have $f: \mathbb{R}^2 \to \mathbb{R}$ (so $n=2$). Set $x = x_1$, $y = x_2$. The formula becomes (w/ $x_0 = (0,0)$)

$$T(h_1,h_2) = f(0,0) + h_1 \frac{\partial f}{\partial x}(0,0) + h_2 \frac{\partial f}{\partial y}(0,0) + \frac{1}{2}\left(h_1^2 \frac{\partial^2 f}{\partial x^2} + h_1 h_2 \frac{\partial^2 f}{\partial x \partial y} + h_2 h_1 \frac{\partial^2 f}{\partial y \partial x} + h_2^2 \frac{\partial^2 f}{\partial y^2}\right)$$

Compute all partials of $f(x,y) = 6e^{x+y}$ at $(0,0)$

Note that $6e^{x+y} = f = f_y = f_x = f_{xy} = f_{yx} = f_{xx} = f_{yy}$

So $T(h_1,h_2) = 6 + 6h_1 + 6h_2 + \frac{1}{2}\left(6h_1^2 + 6h_1h_2 + 6h_2h_1 + 6h_2^2\right)$

is the second order taylor approx. The first order approx. is

$$S(h_1,h_2) = 6 + 6h_1 + 6h_2$$

∎

Problem 5 **Chapter 14.3**

For each function, find all critical points and determine if they are local extema or saddle points: (a) $f(x,y) = x^2 + 2xy + y^2$ (b) $g(x,y) = x\sin y$

Solution

(a) To find the critical points, solve the system of eq: $f_x(x,y)=0$, $f_y(x,y)=0$

In our case
$$2x + 2y = f_x = 0$$
$$2x + 2y = f_y = 0$$

$\Rightarrow$ $x = -y$ so the set of critical points is

$$L = \{(x,-x) : x \in \mathbb{R}\} \quad (\text{the line } y = x)$$

Note that
$$f(x,y) = x^2 + 2xy + y^2 = (x+y)^2$$

If $(x,y) \in L$, then $x = -y$ so
$$f(x,y) = (x-x)^2 = 0.$$

Also $f(x,y) = (x+y)^2 \geq 0$ so every point in L is a global minimum. ∎

(b) $\boxed{g(x,y) = x\sin y}$

Solve
$$\sin y = g_x = 0 \Rightarrow y = \sin^{-1}(0) = k\pi \text{ for } k \in \mathbb{Z}$$
$$x\cos y = g_y = 0 \Rightarrow x = 0 \text{ since } \cos(k\pi) \neq 0.$$

So the set of critical points is
$$C = \{(0, k\pi) : k \in \mathbb{Z}\} \quad (\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\})$$

Theorem 6 Second-Derivative Maximum-Minimum Test for Functions of Two Variables Let $f(x,y)$ be of class C^2 on an open set U in $\mathbb{R}^2$. A point (x_0,y_0) is a (strict) local minimum of f provided the following three conditions hold:

(i) $\frac{\partial f}{\partial x}(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,y_0) = 0$

(ii) $\frac{\partial^2 f}{\partial x^2}(x_0,y_0) > 0$

(iii) $D = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 > 0$ at (x_0,y_0)

(D is called the *discriminant* of the Hessian.) If in (ii) we have < 0 instead of > 0 and condition (iii) is unchanged, then we have a (strict) local maximum.

If $D < 0$ (e.g., if $\frac{\partial^2 f}{\partial x^2}(x_0,y_0) = 0$ or $\frac{\partial^2 f}{\partial y^2}(x_0,y_0) = 0$, but $\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) \neq 0$), then (x_0,y_0) is of *saddle type* (neither a maximum nor a minimum).

(ii) We have
$$g_{xx} = \frac{\partial}{\partial x}\sin y = 0$$

So
$$D = \begin{vmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{vmatrix} = \begin{vmatrix} 0 & \cos y \\ \cos y & -x\sin y \end{vmatrix}$$
$$= 0 - \cos^2 y$$

If $D < 0$ (e.g., if $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) = 0$ or $\frac{\partial^2 f}{\partial y^2}(x_0, y_0) = 0$, but $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \neq 0$), then (x_0, y_0) is of ***saddle type*** (neither a maximum nor a minimum).

$$= 0 - \cos^2 y$$

Evaluate at a critical point $\Rightarrow y = k\pi$

$\Rightarrow \cos y \neq 0$ so $0 - \cos^2 y < 0$

$\Rightarrow$ All critical points are saddle points. $\blacksquare$

Problem 6

Chapter 14.3

Determine the global extreme value of $f(x,y) = 5x^3 - 6y$ on $0 \le x, y \le 2$.