Math 23A, Summer 2019-Session 2 **MIDTERM** UCSC, Tromba & Bäuerle, 8/16/19

Notes: No calculators are allowed. **Please put away your cell phones and other electronic devices, including smart watches, turned off or in airplane mode**. Put your answers in the provided boxes.

Your Name: ______________________ **Your Score:** ____ /100 **Last Initial:** ____

1. (18 points) Let $\vec{u} = (1, 0, 1)$ and $\vec{v} = (0, -1, 1)$. Find the following:

(a) $\vec{u} \cdot \vec{v}$ — 1

(b) $\vec{u} \times \vec{v}$ — $(1,-1,-1)$

(c) $\vec{v} \times \vec{u}$ — $(-1,1,1)$

(d) $\vec{u} \times \vec{u}$ — $(0,0,0)$

(e) $\|\vec{u}\|$ and $\|\vec{v}\|$ — $\|\vec{u}\| =$ $\sqrt{2}$ $\|\vec{v}\| =$ $\sqrt{2}$

(f) Find the area of the parallelogram spanned by the vectors $\vec{u}$ and $\vec{v}$. — $\sqrt{3}$

2. (9 points) Compute the determinants of the following matrices:

(a)

$$\begin{bmatrix} 1 & -3 \\ 1 & 6 \end{bmatrix}$$

(A) 4 (B) 6 (C) 0 (D) -4 (E) 9 — Answer (Letter): E

(b)

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 4 & 2 \\ -1 & 1 & 1 \end{bmatrix}$$

(A) 9 (B) 12 (C) 0 (D) 10 (E) 5 — Answer (Letter): A

(c)

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ -1 & 1 & 1 & 4 \end{bmatrix}$$

Hint: Think before you calculate.

(A) 10 (B) 24 (C) 0 (D) -24 (E) 2 — Answer (Letter): B

1

① a) $u \cdot v = (1,0,1)\cdot(0,-1,1) = 0 + 0 + 1 = 1$

b) $u \times v = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix}$

$= i(0-(-1)) - j(1-0) + k(-1-0)$

$= (1, -1, -1)$

c) $v \times u = -(u \times v) = (-1, 1, 1)$

d) $u \times u = (0,0,0)$ (u is parallel to u)

e) $\|u\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$
$= \|v\|$

f) $\|u \times v\| = \sqrt{3}$ $\square$

② a) $\begin{vmatrix} 1 & -3 \\ 1 & 6 \end{vmatrix} = 1 \cdot 6 - (-3) \cdot 1 = 9$

b) $\begin{vmatrix} 2 & 0 & 1 \\ 1 & 4 & 2 \\ -1 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 4 \\ -1 & 1 \end{vmatrix} = 4 + 5 = 9$

c) $\det(X) = 1 \cdot 2 \cdot 3 \cdot 4 = 24$ $\square$

3. (12 points) Let $\mathcal{P}$ be the plane that contains the points $P(1,0,0), Q(1,1,0)$ and $R(1,1,1)$.

(a) Give a normal vector $\vec{n}$ to the plane $\mathcal{P}$. $\vec{n} =$ $\boxed{(1,0,0)}$

(b) Give an equation of the plane $\mathcal{P}$ in the form $ax + by + cz = d$.

$\boxed{x = 1}$

4. (10 points) For each of the questions below, indicate if the statement is true (T) or false (F).

(a) The determinant of a 3x3 matrix A is nonzero if and only if the matrix A is invertible.

Answer (T/F): $\boxed{T}$

(b) The sum, difference, product and quotient (where the denominator is not zero) of continuous functions are continuous.

Answer (T/F): $\boxed{T}$

(c) The paths $\mathbf{l}_1(t) = (1,-1,-1) + t(1,2,3)$ and $\mathbf{l}_2(t) = (1,-1,-1) + t(-2,-4,-6)$ represent the same line in space.

Answer (T/F): $\boxed{T}$

(d) Let $\vec{v_1}$ and $\vec{v_2}$ be two nonzero vectors in $\mathbb{R}^3$. Then $\vec{v_1} \cdot (\vec{v_1} \times \vec{v_2}) = 0$.

Answer (T/F): $\boxed{T}$

(e) Let A, B be 2x2 matrices. If $det(AB) = 0$, then $det(BA) = 0$.

Answer (T/F): $\boxed{T}$

5. (8 points) Consider the point $P(-1,-1,1)$ in cartesian or rectangular coordinates. Give the cylindrical and spherical coordinates of P.

Cylindrical coordinates of P : $(r,\theta,z) = (\sqrt{2}, \frac{5\pi}{4}, 1)$

Spherical coordinates of P : $(\rho,\theta,\phi) = (\sqrt{3}, \frac{5\pi}{4}$

6. (4 points) The quadratic surface given by $z^2 - 2x^2 - y^2 = 0$ is

(A) an ellipsoid (B) a hyperboloid of one sheet (C) a hyperboloid of two sheets (D) an elliptic cone (E) an elliptic paraboloid (F) None of these

Answer (Letter): $\boxed{D}$

7. (15 points) Consider the path $\mathbf{c}(t) = (t^3, t^2, e^{2t})$. Find the following

(a) The tangent (velocity) vector to this path at $t = 0$:

3) a) Let $u = Q - P = (0,1,0)$, $v = R - P = (0,1,1)$.
Then u, v lie in the plane so $u \times v$ is orthogonal to the plane. So

$$n = u \times v = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = i \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = i = (1,0,0) \quad \square$$

b) $(1,0,0) \cdot ((x,y,z) - (1,0,0)) = 0$

$\Rightarrow x - 1 = 0 \Rightarrow \boxed{x = 1} \quad \square$

4) a) T

b) T

c) Yes, same initial point and direction vectors are parallel.
Note: infinitely many ways to parameterize the same line!

d) True, since $v_1 \times v_2$ is $\perp$ v_1 so the dot product is zero.

e) True, if $\det(AB) = 0$, then

$0 = \det(A) \cdot \det(B) = \det(B) \cdot \det(A) = \det(BA) \quad \square$

5) a) $r = \sqrt{x^2 + y^2} = \sqrt{2}$ $\quad \theta = \begin{cases} \tan^{-1}(\frac{y}{x}), & x > 0 \\ \tan^{-1}(\frac{y}{x}) + \pi, & x < 0 \\ \pm\frac{\pi}{2}, & x = 0 \end{cases} = \tan^{-1}(\frac{-1}{-1}) + \pi = \frac{5\pi}{2}.$

b) $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{3}$ $\quad \theta =$ Same as (a).

$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{3}\right) \quad \square$

6) $z^2 = y^2 + 2x^2 \Rightarrow$ elliptic cone $\square$

7) a) $c'(t) = (3t^2, 2t, 2e^{2t}) \Rightarrow t = 0, c'(0) = (0,0,2) \quad \square$

$c(t) = (t^3, t^2, e^{2t})$, $c'(t) = (3t^2, 2t, 2e^{2t})$

(b) The acceleration vector to this path at $t = 0$: $\boxed{(0,2,4)}$

(c) The parametric equation for the tangent line to this path $\mathbf{c}(t)$ at $t = 0$:

$x = \boxed{0}$ $y = \boxed{0}$ $z = \boxed{2t+1}$

8. (10 points) Consider the path $\mathbf{c}(t) = (\sqrt{7} \cdot t, \sin t - \cos t, \sin t + \cos t)$ where $0 \le t \le 2\pi$.

(a) Set up an integral for the length L of the path $\mathbf{c}(t)$ where $0 \le t \le 2\pi$.

$\boxed{\int_0^{2\pi} 3\,dt}$

(b) Find the arc length L of the path $\mathbf{c}(t)$ where $0 \le t \le 2\pi$.

$L = \boxed{6\pi}$

9. (9 points) Let $f(x, y) = x^3y^3 + x^2 + 1$.

(a) Find $\frac{\partial f}{\partial x}(x, y)$ $\boxed{3x^2y^3+2x}$

(b) Find $\frac{\partial f}{\partial y}(x, y)$ $\boxed{3x^3y^2}$

(c) The equation of the tangent plane to $z = f(x, y)$ at the point $(1, -1)$ is
(A) $z = 1 + (3x^2y^3 + 2x)(x - 1) + (3x^3y^2)(y + 1)$ (B) $z = 1 - (x - 1) + 3(y + 1)$ (C) $z = -x + 3y$ (D) $z = -(x - 1) + 3(y + 1)$ (E) $z = 1 + (x - 1) + 3(y + 1)$ (F) None of these

Answer (Letter): $\boxed{B}$

10. (5 points) Evaluate the limit (if possible) $\lim_{(x,y)\to(0,0)} \frac{xy}{(x + y)^2 - (x - y)^2}$

(A) The limit does not exist (B) $\frac{1}{4}$ (C) -4 (D) $\frac{1}{2}$ (E) 0

Answer (Letter): $\boxed{B}$

b) $c''(t) = (6t, 2, 4e^{2t})$, $t=0$ $c''(0) = (0,2,4)$

c) $c(0) = (0,0,1)$ is a pt in the line. $c'(0) = (0,0,2)$ is a vector in the direction of the line. So an eq is

$$\ell(t) = (0,0,1) + t(0,0,2) = (0,0,2t+1) \quad \square$$

(8) a) $s = \int_0^{2\pi} \|c'(t)\|\,dt = \int_0^{2\pi} \|(\sqrt{7}, \cos t+\sin t, \cos t - \sin t)\|\,dt$

$$= \int_0^{2\pi} \sqrt{7 + (\cos t+\sin t)^2 + (\cos t - \sin t)^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{7 + \underbrace{\cos^2 t + \sin^2 t}_{=1} + \cancel{\sin t\cos t} + \overbrace{\cos^2 t + \sin^2 t}^{=1} - \cancel{\sin t\cos t}}\,dt$$

$$= \int_0^{2\pi} \sqrt{9}\,dt = \int_0^{2\pi} 3\,dt = 6\pi. \quad \square$$

(9) a) $f_x = 3x^2y^3 + 2x$ b) $f_y = 3x^3y^2$

c) $z - \underbrace{f(1,-1)}_{1} = \underbrace{f_x(1,-1)}_{-1}(x-1) + \underbrace{f_y(1,-1)}_{3}(y+1)$

$$z = 1 - (x-1) + 3(y+1) \quad \square$$

(10) $\lim_{(x,y)\to(0,0)} \frac{xy}{(x+y)^2 - (x-y)^2} = \lim \frac{xy}{(x+y+(x-y))(x+y-(x-y))}$

$$= \lim_{(x,y)\to(0,0)} \frac{xy}{(2x)2y} = \lim_{(x,y)\to(0,0)} \frac{1}{4} = \frac{1}{4} \quad \square$$