Problem 1

Chapter 12.2

Consider the helix given by $c(t) = (a\cos t, a\sin t, bt)$. Show that the acceleration $a(t)$ is always parallel to the xy-plane.

Solution We need to show that $a(t) = c''(t) \perp (0,0,1)$.

We have

$$v(t) = c'(t) = (-a\sin t, a\cos t, b)$$

$$a(t) = v'(t) = (-a\cos t, -a\sin t, 0)$$

To see that $a(t) \perp (0,0,1)$ we compute

$$a(t)\cdot(0,0,1) = (-a\cos t, -a\sin t, 0)\cdot(0,0,1)$$
$$= 0.$$

$\blacksquare$

Problem 2 | Chapter 12.2

The acceleration, initial velocity, and initial position of a particle are given by $a(t) = (-6, 2, 4)$, $v(0) = (2, -5, 1)$, and $r(0) = (-3, 6, 2)$.
When does the particle intersect the yz plane?

Solution First, we need to compute the position function for the particle using the fact that $r''(t) = a(t)$

We have

$$v(t) = \int a(t)\,dt = \left(\int -6, \int 2, \int 4 \right)$$
$$= (-6t + a, 2t + b, 4t + c)$$
$$= (-6t, 2t, 4t) + (a, b, c)$$

Then $(2, -5, 1) = v(0) = (0, 0, 0) + (a, b, c)$ so $(a, b, c) = (2, -5, 1)$. So $v(t) = (-6t + 2, 2t - 5, 4t + 1)$.

Next

$$r(t) = \int v(t)\,dt = \left(\int -6t+2, \int 2t-5, \int 4t+1 \right)$$
$$= (-3t^2 + 2t, t^2 - 5t, 2t^2 + t) + (x, y, z)$$

$(-3, 6, 2) = r(0) = (0, 0, 0) + (x, y, z)$ so $(x, y, z) = (-3, 6, 2)$

So $r(t) = (-3t^2 + 2t - 3, t^2 - 5t + 6, 2t^2 + t + 2)$

An equation of the yz plane is $x = 0$ so we set

$$x(t) = -3t^2 + 2t - 3 = 0 \Rightarrow t = \frac{-2 \pm \sqrt{4 - 4\cdot(-3)(-3)}}{-6}$$
$$= \frac{-2 \pm \sqrt{4 - 36}}{-6}$$
$$= 1/3 \pm i\sqrt{32}$$

Since the values of t are not real numbers, the particle does not intersect the yz-plane.

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$\boxed{\text{Problem 3}}$ $\boxed{\text{Chapter 12.3}}$

Find the arc length of $c(t) = (t, |t|)$ for $-1 \le t \le 1$.

<u>Solution</u> Arclength s of a curve $c(t)$ for $a \le t \le b$

$$s = \int_a^b \|c'(t)\|\, dt$$

Notice that $|t|$ is not differentiable when $t=0$.

Note that $|t| = \begin{cases} t, & t \ge 0 \\ -t, & t < 0 \end{cases}$

We have

$$s = \int_{-1}^{1} \|c'(t)\|\, dt = \int_{-1}^{0} \|c'(t)\| dt + \int_0^1 \|c'(t)\|\, dt$$

$$= \int_{-1}^{0} \|(t, -t)'\| dt + \int_0^1 \|(t,t)'\| dt$$

$$= \int_{-1}^{0} \sqrt{(1)^2 + (-1)^2}\, dt + \int_0^1 \sqrt{1^2 + 1^2}\, dt$$

$$= \int_{-1}^{0} \sqrt{2}\, dt + \int_0^1 \sqrt{2}\, dt$$

$\int_a^x f + \int_x^b f = \int_a^b f \longrightarrow$

$$= \int_{-1}^{1} \sqrt{2}\, dt$$

$$= \sqrt{2}\,(1 - (-1)) = 2\sqrt{2}.$$

∎

Problem 4 | Chapter 12.3

Let c be the path $c(t) = (2t, t^2, \log t)$ for $t > 0$. Find the arc length between the points $(2,1,0)$ and $(4,4,\log 2)$.

Solution Recall $s = \int_a^b \|c'(t)\|\, dt$ for $a \leq t \leq b$

Note that $(2,1,0) = c(1)$ and $(4,4,\log 2) = c(2)$

So we compute

$$s = \int_1^2 \|c'(t)\|\, dt = \int_1^2 \|(2, 2t, \tfrac{1}{t})\|\, dt$$

$$= \int_1^2 \sqrt{4 + 4t^2 + \tfrac{1}{t^2}}\, dt$$

We can factor $4t^2 + 4 + \frac{1}{t^2} = \left(2t + \frac{1}{t}\right)^2$

So we have

$$= \int_1^2 \sqrt{\left(2t + \tfrac{1}{t}\right)^2}\, dt$$

$$= \int_1^2 2t + \frac{1}{t}\, dt$$

$$= t^2 + \log t \Big|_1^2$$

$$= (2^2 + \log 2) - (1^2 + \log 1)$$

$$= 3 + \log 2$$

$\blacksquare$

$\boxed{\text{Problem 5}}$ $\boxed{\text{Chapter 12.2}}$

Let $c(t)$ be a path, $v(t)$ its velocity, and $a(t)$ the acceleration. Suppose that $\underline{F(c(t)) = m\,a(t)}$ for some $m > 0$. Show that

$$\frac{d}{dt}[m\,c(t) \times v(t)] = c(t) \times F(c(t))$$

What can we say if F and c are parallel?

<u>Solution</u> Recall $\frac{d}{dt}\left[a(t) \times b(t)\right] = a'(t) \times b(t) + a(t) \times b'(t)$

We have
$$\frac{d}{dt}\Big[\underbrace{(m\,c(t)) \times v(t)}_{\text{angular momentum}}\Big] = (m\,c'(t)) \times v(t) + (m\,c(t)) \times v'(t)$$
$$= (m\,v(t)) \times v(t) + (m\,c(t)) \times a(t)$$

$\left(\vec{v} \times (c\vec{w}) = (c\vec{v}) \times \vec{w} = c(\vec{v} \times \vec{w})\right)$
$$= m(v(t) \times v(t)) + c(t) \times (m\,a(t))$$

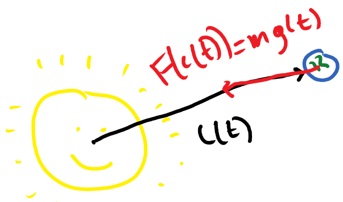
$(v(t) \times v(t) = 0)$
$$= m \cdot 0 + c(t) \times F(c(t))$$
$$= \underbrace{c(t) \times F(c(t))}_{\text{torque}}$$

This equation says

rate of change of angular momentum = torque

What can we say if F and c are parallel? We know that $c(t) \times F(c(t)) = 0$ when F and c are parallel. This tells us that $\frac{d}{dt}\left[(m\,c(t)) \times v(t)\right] = 0$. So $(m\,c(t)) \times v(t)$ is constant.

<u>Ex</u>



The position of the earth and the Force of the suns gravity exerted on the earth are parallel. So the angular momentum of the earths rotation is constant.