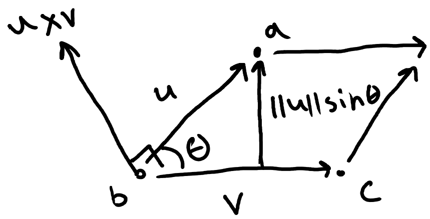
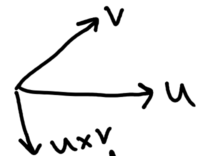
Problem 1 | Chapter 11.3

Determine whether the points (1,1,1), (3,2,3), and (-3,-1,3) are collinear. (Problem 5 from Wed)





<u>Def</u> The cross product $u \times v$ with magnitude given by

$$\|u \times v\| = \|u\|\|v\| \sin\theta = \text{area of the parallelagram}$$

- is perpindicular to both u and v
- satisfies the right hand rule.

<u>Solution</u> The three points are collinear if and only if the parallelagram spanned by the vectors

$$u = (1,1,1) - (3,2,3) = (-2,-1,-2), \qquad v = (1,1,1) - (-3,-1,3) = (4,2,-2)$$

has zero area i.e. $\|u \times v\| = 0$

we have

$$u \times v = \begin{array}{ccc} + & - & + \end{array}$$
$$u \times v = \begin{vmatrix} i & j & k \\ -2 & -1 & -2 \\ 4 & 2 & -2 \end{vmatrix}$$

$$= i\begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix} - j\begin{vmatrix} -2 & -2 \\ 4 & -2 \end{vmatrix} + k\begin{vmatrix} -2 & -1 \\ 4 & 2 \end{vmatrix}$$

$$= i(2-(-4)) - j(4-(-8)) + k(-4-(-4))$$

$$= (6, -12, 0)$$

Thus $\|u \times v\| \neq 0$ so the points are not collinear.

★ During section I thought I made a mistake, but the three points are actually not collinear!

Problem 2 **Chapter 11.3**

Let $u, v, w \in \mathbb{R}^3$. Suppose that there are scalars $\alpha, \beta \in \mathbb{R}$ such that $u = \alpha v + \beta w$. Compute the value of the scalar triple product $u \cdot (v \times w)$

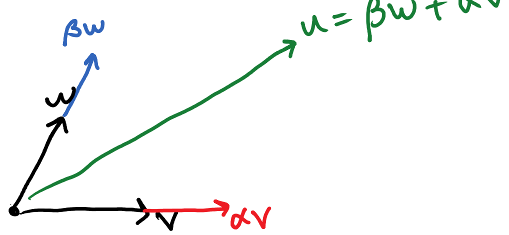
Solution

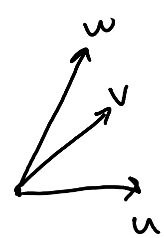
$$
\begin{aligned}
u \cdot (v \times w) &= (\alpha v + \beta w) \cdot (v \times w) \\
&= (\alpha v) \cdot (v \times w) + (\beta w) \cdot (v \times w) \\
&= \alpha \big(\underbrace{v \cdot (v \times w)} \big) + \beta \big(w \cdot (v \times w) \big)
\end{aligned}
$$

Recall
- $v \times w$ is orthogonal to v
- $a \cdot b = 0$ if and only if $a \perp b$

$$
\begin{aligned}
&= \alpha \cdot 0 + \beta \cdot 0 \\
&= 0
\end{aligned}
$$

Geometrically

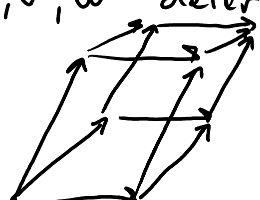




u, v, w are linearly dependent

u, v, w are linearly independent

u, v, w determines a parallelepiped:



Fact: Volume of parallelepiped is given by $u \cdot (v \times w)$

Signed

Also, $u \cdot (v \times w) = \det \begin{bmatrix} u \\ v \\ w \end{bmatrix}$

Problem 3 | Chapter 11.3

Determine whether the vectors $\underbrace{(1,4,-7)}_{u}$, $\underbrace{(2,-1,4)}_{v}$, and $\underbrace{(0,-9,18)}_{w}$ are coplanar.

Solution By Problem 2, we just need to compute $u\cdot(v\times w)$.

We have

$$u\cdot(v\times w) = \det\begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$= \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 2\begin{vmatrix} 4 & -7 \\ -9 & 18 \end{vmatrix} + 0\begin{vmatrix} 4 & -7 \\ -1 & 4 \end{vmatrix}$$

$$= -18-(-36) - 2(72-(63))$$

$$= 18 - 18 = 0$$

Since $u\cdot(v\times w) = 0$, the vectors lie in the same plane!

$\blacksquare$

Ex Computing a determinant:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} = -0\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix}$$

$$= 0 + 0 - 1(0)$$

$$= 0$$

Fact $\det(A) = 0$ if there is a row or column containing only zeros.

Fact $\det(A) = 0$ if one row/column is a multiple of another row/column.

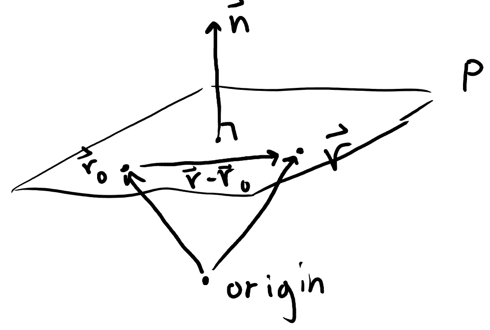
Problem 4 | Chapter 11.3

Consider the lines $r_1(t) = (1,1,0) + t(1,-1,2)$ and $r_2(S) = (2,0,2) + S(-1,1,0)$

a) Show that r_1, r_2 intersect.

b) Find an equation of the plane containing r_1 and r_2.

How do we describe a plane?



Given a point $\vec{r}_0$ in the plane P and a vector $\vec{n}$ orthogonal to P, a point $\vec{r}$ is in P if and only if $\vec{n} \cdot \vec{r} - \vec{r}_0 = 0$

So the collection of point in P are the points $\vec{r}$ that satisfy the equation

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \qquad (*)$$

If $n = (a,b,c)$, $\vec{r} = (x,y,z)$, $r_0 = (x_0, y_0, z_0)$, then $(*)$ becomes $\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$

Solution (a) $r_1(t) = (1,1,0) + t\underline{(1,-1,2)}$ and $r_2(S) = (2,0,2) + S\underline{(-1,1,0)}$

Set $r_1(t) = r_2(s)$: $(1+t, 1-t, 2t) = (2-s, s, 2)$

$$\Rightarrow \begin{cases} 1+t = 2-s \\ 1-t = s \\ 2t = 2 \end{cases} \; (3)$$

So $t = 1$ by (3)

Then (1) implies $s = 0$

(2) implies $s = 0$.

So the system has a solution, $t=1$, $s=0$. So the lines intersect at the point $(2,0,2)$.

(b) A point in the plane is $(2,0,2)$. We need to find a normal vector. Since the vectors $(1,-1,2)$ and $(-1,1,0)$ are parallel to the plane, we can $(1,-1,2) \times (-1,1,0)$ as the normal vector.

We have

$$n = (1,-1,2) \times (-1,1,0) = \begin{vmatrix} i & j & +k \\ 1 & -1 & -2 \end{vmatrix}$$

$$n = (1,-1,2) \times (-1,1,0) = \begin{vmatrix} i & j & k \\ 1 & -1 & -2 \\ -1 & 1 & +0 \end{vmatrix}$$

$$= k\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} i & j \\ -1 & 1 \end{vmatrix}$$

$$= 0 \cdot k - 2(i+j) = (-2,-2,0).$$

So the equation of the plane is

$$(-2,-2,0)\cdot\big((x,y,z) - (2,0,2)\big) = 0$$

$$\Rightarrow \boxed{-2(x-2) - 2y = 0}$$