Problem 1

Determine whether the points \((1,1,1)\), \((3,2,3)\), and \((-3,-1,3)\) are collinear. (Problem 5 from Wed)

**Definition** The cross product \(u \times v\) with magnitude given by

\[ |u \times v| = |u| |v| \sin \theta = \text{area of the parallelogram} \]

- is perpendicular to both \(u\) and \(v\)
- satisfies the right hand rule.

**Solution** The three points are collinear if and only if the parallelogram spanned by the vectors

\[ u = (1,1,1) - (3,2,3), \quad v = (3,2,3) - (-3,-1,3) \]
\[ = (-2,-1,-2) \quad = (4,2,2) \]

has zero area, i.e. \(|u \times v| = 0\).

We have

\[ u \times v = \begin{vmatrix} i & j & k \\ -2 & -1 & -2 \\ 4 & 2 & -2 \end{vmatrix} \]
\[ = i \begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix} - j \begin{vmatrix} -2 & -2 \\ 4 & -2 \end{vmatrix} + k \begin{vmatrix} -2 & -1 \\ 4 & 2 \end{vmatrix} \]
\[ = i((-1)(-2) - (-2)(2)) - j((-2)(-2) - (-2)(4)) + k((-2)(2) - (-1)(4)) \]
\[ = (6, -12, 0) \]

Thus \(|u \times v| = 0\) so the points are not collinear.

*During section I thought I made a mistake, but the three points are actually not collinear.*
Problem 2

Let \( u, v, w \in \mathbb{R}^3 \). Suppose that there are scalars \( \alpha, \beta \in \mathbb{R} \) such that \( u = \alpha v + \beta w \). Compute the value of the scalar triple product \( u \cdot (v \times w) \).

Solution

\[
\begin{align*}
 u \cdot (v \times w) &= (\alpha v + \beta w) \cdot (v \times w) \\
 &= (\alpha v) \cdot (v \times w) + (\beta w) \cdot (v \times w) \\
 &= \alpha (v \cdot (v \times w)) + \beta (w \cdot (v \times w)) \\
\end{align*}
\]

Recall, \( v \times w \) is orthogonal to \( v \).

\( \cdot a \cdot b = 0 \) if and only if \( \alpha \perp \beta \)

\[
= \alpha \cdot 0 + \beta \cdot 0 = 0
\]

Geometrically

\( u, v, w \) are linearly dependent \( \rightarrow u, v, w \) are linearly independent

\( u, v, w \) determines a parallelepiped:

Fact: Volume of parallelepiped is given by \( u \cdot (v \times w) \)

Also, \( u \cdot (v \times w) = \det \begin{vmatrix} u & v \\ v & w \end{vmatrix} \)
Solution By Problem 2, we just need to compute \( u \cdot (v \times w) \).

We have

\[
\begin{align*}
\quad u \cdot (v \times w) &= \det \begin{bmatrix} u & v \\ w \\ \end{bmatrix} \\
&= \begin{vmatrix}
1 & 4 & -7 \\
2 & -1 & 4 \\
0 & -9 & 18 \\
\end{vmatrix} \\
&= 1 \begin{vmatrix} -1 & -7 \\ -9 & 18 \end{vmatrix} - 2 \begin{vmatrix} 4 & -7 \\ -9 & 18 \end{vmatrix} + 0 \begin{vmatrix} 4 & -7 \\ -9 & 18 \end{vmatrix} \\
&= -18 - (-36) - 2(72 - (63)) \\
&= 18 - 18 = 0
\end{align*}
\]

Since \( u \cdot (v \times w) = 0 \), the vectors lie in the same plane.

Ex. Computing a determinant:

\[
\begin{bmatrix}
1 & 0 & 0 \\
1 & 0 & 2 \\
0 & 1 & 1 \\
\end{bmatrix} = -0 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} + 0 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} \\
= 0 + 0 -1(0) \\
= 0
\]

Fact \( \det(A) = 0 \) if there is a row or column containing only zeros.
Fact \( \det(A) = 0 \) if one row/column is a multiple of another row/column.
Problem 4

Consider the lines \( r_1(t) = (1,1,0) + t(1,-1,2) \) and \( r_2(s) = (2,0,2) + s(-1,1,0) \)

a) Show that \( r_1, r_2 \) intersect.

b) Find an equation of the plane containing \( r_1 \) and \( r_2 \).

How do we describe a plane?

Given a point \( \bar{r}_0 \) in the plane \( P \) and a vector \( \hat{n} \) orthogonal to \( P \), a point \( \bar{r} \) is in \( P \) if and only if \( \hat{n} \cdot (\bar{r} - \bar{r}_0) = 0 \)

So the collection of points in \( P \) are the points \( \bar{r} \) that satisfy the equation

\[
\hat{n} \cdot (\bar{r} - \bar{r}_0) = 0
\]

If \( \hat{n} = (a,b,c) \), \( \bar{r} = (x,y,z) \), \( \bar{r}_0 = (x_0,y_0,z_0) \), then (4) becomes

\[
a(x-x_0) + b(y-y_0) + c(z-z_0) = 0
\]

Solution (a) \( r_1(t) = (1,1,0) + t(1,-1,2) \) and \( r_2(s) = (2,0,2) + s(-1,1,0) \)

Set \( r_1(t) = r_2(s) \) : \( (1+t, 1-t, 2t) = (2-s, s, 2) \)

\[
\begin{align*}
1+t &= 2-s \\
1-t &= s \\
2t &= 2
\end{align*}
\]

So \( t = 1 \) by (3)

Then (1) implies \( s = 0 \)

(2) implies \( s = 0 \).

So the system has a solution, \( t = 1, s = 0 \). So the lines intersect at the point \( (2,0,2) \).

(b) A point in the plane is \( (2,0,2) \). We need to find a normal vector.

Since the vectors \((1,1,2)\) and \((-1,1,0)\) are parallel to the plane, we can \((1,1,2) \times (-1,1,0)\) as the normal vector.

\[
N = (1,1,2) \times (-1,1,0) = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{vmatrix}
\]

\[
= (1 \cdot 0 - 2 \cdot 1) i - (1 \cdot (-1) - 2 \cdot 1) j + (1 \cdot 1 - 1 \cdot (-1)) k
\]

\[
= -2i + 3j + 2k
\]
\[ \mathbf{n} = (1, -1, 2) \times (-1, 1, 0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{vmatrix} \]

\[ = \mathbf{k} \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} - 2 \begin{vmatrix} \mathbf{i} & \mathbf{j} \\ -1 & 1 \end{vmatrix} \]

\[ = 0 \mathbf{k} - 2(\mathbf{i} + \mathbf{j}) = (-2, -2, 0). \]

So the equation of the plane is

\[ (-2, -2, 0) \cdot ((x, y, z) - (2, 1, 2)) = 0 \]

\[ \Rightarrow -2(x-2) - 2y = 0 \]