Problem 1

Tuesday, May 26, 2020 3:05 PM

$\boxed{14.3}$

① For each function, find all critical points and determine whether they are local maxima / minima or saddle points

a) $f(x,y) = x^2 + 2xy + y^2$
b) $g(x,y) = x \sin y$

Solution (a) The critical points are the points (x,y) such that

$$\begin{cases} 0 = f_x = 2x + 2y \\ 0 = f_y = 2x + 2y \end{cases}$$

So the critical all lie on the line $y = -x$, i.e., the collection of critical points is

$$\{(x,-x) \mid x \in \mathbb{R}\},$$

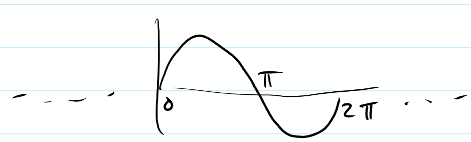
The second derivitive test fails! But we can factor

$$f(x,y) = x^2 + 2xy + y^2$$
$$= (x+y)^2$$

So $f(x,y) \geq 0$ for any (x,y). For any critical point (x,y) we have $f(x,y) = f(x,-x) = (x-x)^2 = 0$. This shows that every critical point is a minimum.

(b) Let $g(x,y) = x \sin y$. The critical points (x,y) are the solutions to the system of eq's

$$\begin{cases} 0 = f_x = \sin y \\ 0 = f_y = x \cos y \end{cases}$$



From $\sin y = 0$ we know $y = k\pi$, $k \in \mathbb{Z}$. Since $\cos k\pi \neq 0$ we obtain $x = 0$. So the critical points are

$$\{(0, k\pi) \mid k \in \mathbb{Z}\}$$

The discriminant of the Hessian is < 0 since $f_{xx}(0, k\pi) = 0$ independent of $k \in \mathbb{Z}$. Thus, every critical point is a saddle point by the 2nd-Derivative test. $\blacksquare$

Problem 2

Tuesday, May 26, 2020 3:52 PM

$\boxed{14.4}$

② Find the shortest distance between the point $(1,0,-1)$ and the plane $2x-2y+2z=6$.

<u>Solution</u> We need to minimize $d(x,y,z)=\sqrt{(x-1)^2+y^2+(z+1)^2}$ subject to the constraint $\underbrace{2x-2y+2z}_{g(x,y,z)}=6$. Notice any minimum point for $d^2(x,y,z)=(x-1)^2+y^2+(z+1)^2$ will also be a minimum for $d(x,y,z)$. By Lagrange multiplier Theorem, if d^2 attains a max or min on $2x-2y+2z=6$, then there is $\lambda\in\mathbb{R}$ such that

$$(2x-2,2y,2z+2)=\nabla d^2=\lambda\nabla g=\lambda(2,-2,2)$$

So the critical points are the solutions to

$$\begin{cases} 2x-2=2\lambda & \Rightarrow x=\lambda+1 \\ 2y=-2\lambda & \Rightarrow y=-\lambda \\ 2z+2=2\lambda & \Rightarrow z=\lambda-1 \\ 2x-2y+2z=6 & \Rightarrow 2(\lambda+1)-2(-\lambda)+2(\lambda-1)=6 \end{cases}$$

$$\Rightarrow 6\lambda=6 \Rightarrow \lambda=1$$

So $x=2$, $y=-1$, $z=0$, i.e., $(2,-1,0)$ is the only critical point. Apply the second derivative test: define $h=d^2-\lambda g$
$=d^2-g$.

Find the determinant

$$|\bar{H}|=\begin{vmatrix} 0 & -g_x & -g_y & -g_z \\ -g_x & h_{xx} & h_{yx} & h_{zx} \\ -g_y & h_{yx} & h_{yy} & h_{zy} \\ -g_z & h_{zx} & h_{zy} & h_{zz} \end{vmatrix}$$

$g=2x-2y+2z$

$d^2=(x-1)^2+y^2+(z+1)^2$

If $|\bar{H}|>0 \Rightarrow$ max

$|\bar{H}|<0 \Rightarrow$ min

$|\bar{H}|=0 \Rightarrow$ inconclusive.

$|\, 0 \;\; -2 \;\; 2 \;\; -2 \,|$

$$= \begin{vmatrix} 0 & -2 & 2 & -2 \\ -2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{vmatrix} = -48 < 0 \Rightarrow (2,-1,0) \text{ is a minimum.}$$

Problem 3

Tuesday, May 26, 2020 6:20 PM

(3) Let P be a point on the surface S in $\mathbb{R}^3$ defined by the equation $f(x,y,z)=1$, where f is continuously differentiable. Suppose the distance between S and $(0,0,0)$ is maximized at P. Show that the vector emanating from $(0,0,0)$ and ending at P is orthogonal to S.

(4) Let A be a non-zero symmetric 3×3 matrix. Define $f:\mathbb{R}^3\to\mathbb{R}$ via

$$f\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \frac{1}{2}\Big(\underbrace{A\begin{bmatrix}x\\y\\z\end{bmatrix}}_{3\times 1}\Big)\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}$$

dot product

(a) Find ∇f.

(b) Restrict f to the unit sphere S. Does f achieve a global max/min?

(c) Show that there exists a point $x\in S$ and $\lambda\neq 0$ such that

$$Ax = \lambda x.$$

Solution (a) Write $A = \begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{12} & a_{22} & a_{23}\\ a_{13} & a_{23} & a_{33}\end{bmatrix}$. Then

$$f\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \frac{1}{2}\left(a_{11}x^2 + a_{12}xy + a_{13}xz + a_{12}xy + a_{22}y^2 + a_{23}yz + a_{13}xz + a_{23}yz + a_{33}z^2\right)$$

Then $\nabla f = \begin{bmatrix} f_x\\ f_y\\ f_z\end{bmatrix}$ so we compute

$$f_x = \frac{1}{2}(2a_{11}x + 2a_{12}y + 2a_{13}z) = a_{11}x + a_{12}y + a_{13}z$$

$$f_y = \frac{1}{2}(2a_{12}x + 2a_{22}y + 2a_{23}z) = a_{12}x + a_{22}y + a_{23}z$$

$$f_z = a_{13}x + a_{23}y + a_{33}z$$

So, $\nabla f = A\begin{bmatrix}x\\y\\z\end{bmatrix}$. ☒

(b) Does f achieve a max/min on $S = \{(x,y,z) \mid x^2+y^2+z^2=1\}$?

Yes, f is continous since it is a polynomial and S is closed and bounded. ☒

(c) Show that there exists a point $x\in S$ and $\lambda\neq 0$ such that $Ax=\lambda x$.

By (b), f achieves a max at some point $(x_0,y_0,z_0)\in S$.

By the Lagrange multipliers, there exist $\alpha\in\mathbb{R}$ such that

$$A\begin{bmatrix}x_0\\y_0\end{bmatrix} = \nabla f(x_0,y_0,z_0) = \alpha\nabla g(x_0,y_0,z_0) \quad \text{where } g = x^2+y^2+y^2$$

$$A\begin{bmatrix}x_0\\y_0\\z_0\end{bmatrix} = \nabla f(x_0,y_0,z_0) = \alpha \nabla g(x_0,y_0,z_0) \quad \text{where } g = x^2+y^2+y^2$$

$$= \alpha \begin{bmatrix}2x_0\\2y_0\\2z_0\end{bmatrix}$$

$$= 2\alpha \begin{bmatrix}x_0\\y_0\\z_0\end{bmatrix}$$

So take $\lambda = 2\alpha$. the $\underline{A}\begin{bmatrix}x_0\\y_0\\\underline{z_0}\end{bmatrix} = \lambda\begin{bmatrix}x_0\\y_0\\z_0\end{bmatrix}$ and $\lambda \neq 0$ since $\begin{bmatrix}x_0\\y_0\\z_0\end{bmatrix} \neq 0$ as $x_0^2 + y_0^2 + z_0^2 = 1$ and $A \neq 0$. this proves the claim.

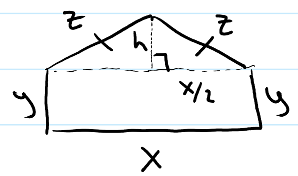
We proved a theorem in linear algebra: every real symmetric matrix has a non-zero eigenvalue.

$\blacksquare$

Friday, May 29, 2020 5:07 PM

> Suppose that a pentagon is composed of a rectangle topped by an isosceles triangle. If the length of the perimeter is fixed, find the maximum possible area. Assume that the sides of the triangle that are required to be same length do not share a side with the rectangle.
>
> From <https://cole2.uconline.edu/courses/1294037/assignments/13753524>

The picture



$$h = \sqrt{z^2 - \frac{x^2}{4}}$$

If the perimeter is fixed, then $\underbrace{x + 2y + 2z}_{g} = K$ for $K \in \mathbb{R}$.

We need to maximize

$$A(x,y,z) = xy + \frac{1}{2}xh$$

$$= xy + \frac{1}{2}x\sqrt{z^2 - \frac{x^2}{4}}$$

By Lagrange $\exists \lambda \in \mathbb{R}$ s.t.

$$\left(y + \frac{1}{2}\sqrt{z^2 - \frac{x^2}{4}} - \frac{x}{4\sqrt{z^2-\frac{x^2}{4}}}\frac{1}{2}x,\ x,\ \frac{1}{4}x \cdot \frac{2z}{\sqrt{z^2 - \frac{x^2}{4}}}\right) = \nabla A = \lambda \nabla g = \lambda(1,2,2)$$

So we have

$$\begin{cases} y + \frac{1}{2}\sqrt{z^2 - \frac{x^2}{4}} - \frac{x^2}{8\sqrt{z^2 - \frac{x^2}{4}}} = \lambda \\ x = 2\lambda \\ \frac{1}{2}\frac{xz}{\sqrt{z^2 - \frac{x^2}{4}}} = 2\lambda \\ x + 2y + 2z = K \end{cases}$$

$$x = 2\lambda \implies \frac{\lambda z}{\sqrt{z^2 - \lambda^2}} = 2\lambda$$

$$\implies \lambda z = 2\lambda\sqrt{z^2 - \lambda^2} \qquad (\text{note } \lambda > 0)$$

$$\implies z = 2\sqrt{z^2 - \lambda^2}$$

$$\Rightarrow \quad z = 2\sqrt{z^2-\lambda^2}$$

$$\Rightarrow \quad z^2 = 4z^2 - 4\lambda^2$$

$$\Rightarrow \quad 3z^2 = 4\lambda^2$$

$$\Rightarrow \quad \boxed{z = \frac{2}{\sqrt{3}}\lambda}.$$

Then $$y = \lambda + \frac{x^2}{8\sqrt{z^2-\frac{x^2}{4}}} - \frac{1}{2}\sqrt{z^2-\frac{x^2}{4}}$$

$$= \lambda + \frac{4\lambda^2}{8\sqrt{\frac{4}{3}\lambda^2-\lambda^2}} - \frac{1}{2}\sqrt{\frac{4}{3}\lambda^2-\lambda^2}$$

$$= \lambda + \frac{1}{2}\frac{\lambda}{\sqrt{\frac{1}{3}}} - \frac{1}{2}\lambda\sqrt{\frac{1}{3}}$$

$$= \lambda\left(1+\frac{1}{2}\cdot\frac{\sqrt{3}}{1} - \frac{1}{2}\frac{\sqrt{3}}{3}\right)$$

$$= \lambda\left(1+\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{6}\right)$$

So $$K = x+2y+2z = 2\lambda + 2\lambda\left(1+\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{6}\right) + \frac{4\sqrt{3}}{3}\lambda$$

So $$\lambda = \frac{K}{2+2+\sqrt{3}-\frac{\sqrt{3}}{3}+\frac{4\sqrt{3}}{3}} = \frac{K}{4+\frac{6\sqrt{3}}{3}} = \frac{K}{4+2\sqrt{3}}$$

So $$\boxed{\begin{aligned} x &= 2\left(\frac{K}{4+2\sqrt{3}}\right) = \frac{K}{2+\sqrt{3}} \qquad y = \left(1+\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{6}\right)\left(\frac{K}{4+2\sqrt{3}}\right) \\ z &= \frac{2\sqrt{3}}{3}\left(\frac{K}{4+2\sqrt{3}}\right) \end{aligned}}$$