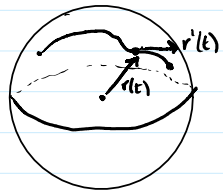
Problem 1

Monday, April 20, 2020 4:58 AM

① Let $r(t)$ be a parameterization of a curve. Suppose that $\|r(t)\| = c$ for all $t \in \mathbb{R}$. Show that $r(t)$ and $r'(t)$ are orthogonal for all $t \in \mathbb{R}$.



$\|r(t)\| = c \iff r(t)$ is a collection of points on the sphere of radius c centered at the origin

<u>Proof</u> We need to show $r(t) \cdot r'(t) = 0$ for any $t \in \mathbb{R}$. We know $c^2 = \|r(t)\|^2$.

Hence,

$$0 = \frac{d}{dt} c^2 = \frac{d}{dt} \|r(t)\|^2$$

$$= \frac{d}{dt} \left(r(t) \cdot r(t) \right) \qquad (v \cdot w)' = v'w + vw'$$

$$= r'(t) \cdot r(t) + r(t) \cdot r'(t)$$

$$= 2 r(t) \cdot r'(t).$$

Dividing by 2, we obtain $r(t) \cdot r'(t) = 0$. ∎

<u>Definitions</u> Given any parameterized curve $r(t)$, we can always define the <u>Unit Tangent Vector</u>

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

Note that $\|T(t)\| = 1$ and $T(t)$ is tangent to $r(t)$.

By Problem ①, $T(t) \cdot T'(t) = 0$ for all $t \in \mathbb{R}$. So, we define the <u>Unit Normal Vector</u>

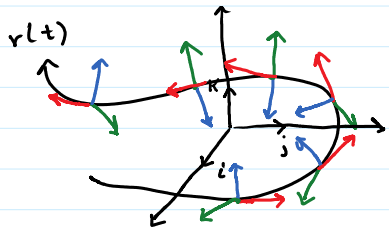
$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

Note that $\|N(t)\| = 1$ and N is orthogonal to $T(t)$. In fact, $N(t)$ points in the direction of local concavity of $r(t)$.

Finally, we define the <u>Unit Binormal Vector</u>

$$B(t) = T(t) \times N(t).$$

Note $\|B(t)\| = 1$ and B is orthogonal to both T and N.



$T(t)$

$N(t)$

$B(t) = T(t) \times N(t)$

Right hand rule!

Note that for each $t \in \mathbb{R}$ we can obtain $\{T(t), N(t), B(t)\}$ by translating and rotating $\{i, j, k\}$. In linear algebra, this mean $\{T(t), N(t), B(t)\}$ is an orthonormal basis for $\mathbb{R}^3$. In physics this is referred to as the TNB-Frame for $r(t)$.

Problem 2

Monday, April 20, 2020 5:28 AM

Def Let $r(t)$ be a parameterized space curve and P a point on the curve.

- The <u>Normal plane at P</u> is the plane containing P, N, & B.
- The <u>Osculating Plane at P</u> is the plane containing P, T, & N.

Recall

For a parameterized curve $r(t)$ we have

$T(t) = \dfrac{r'(t)}{\|r'(t)\|}$ — unit tangent vector

$N(t) = \dfrac{T'(t)}{\|T'(t)\|}$ — unit normal vector

$B(t) = T(t) \times N(t)$ — unit binormal vector

② The helix is given by $r(t) = (\cos t, \sin t, t)$. When $t = \pi/2$, find:

(a) The Normal plane

(b) The Osculating plane.

<u>Solution</u> (a) When $t = \frac{\pi}{2}$, $r(\pi/2) = (\cos \pi/2, \sin \pi/2, \pi/2)$

$$= (0, 1, \pi/2)$$

If N and B are contained in the plane, then any vector orthogonal to the plane must be parallel to $N(\pi/2) \times B(\pi/2)$. By definition, $r'(t)$ is orthogonal to the plane. We have

$$r'(\pi/2) = (-\sin \pi/2, \cos \pi/2, 1)$$
$$= (-1, 0, 1)$$

So, the eq. of the plane is

$$-x + z - \pi/2 = (-1, 0, 1) \cdot (x - 0, y - 1, z - \pi/2) = 0$$

or equivalently $\boxed{x - z = -\pi/2}$

(b) We need to compute $T(t)$, $N(t)$ and $B(t)$.

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{(-\sin t, \cos t, 1)}{\sqrt{\sin^2 t + \cos^2 t + 1}} = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1)$$

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{\frac{1}{\sqrt{2}}(-\cos t, -\sin t, 0)}{\frac{1}{\sqrt{2}}\sqrt{\sin^2 t + \cos^2 t}} = (-\cos t, -\sin t, 0)$$

$$B(t) = T(t) \times N(t) = \begin{vmatrix} i & j & k \\ -\frac{\sin t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} & 1/\sqrt{2} \\ -\cos t & -\sin t & 0 \end{vmatrix} \qquad \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$= -\cos t \begin{vmatrix} j & k \\ \frac{\cos t}{\sqrt{2}} & 1/\sqrt{2} \end{vmatrix} - (-\sin t) \begin{vmatrix} i & k \\ -\frac{\sin t}{\sqrt{2}} & 1/\sqrt{2} \end{vmatrix}$$

$$= -\cos t \left(1/\sqrt{2}\, j - \frac{\cos t}{\sqrt{2}} k \right) + \sin t \left(\frac{1}{\sqrt{2}} i + \frac{\sin t}{\sqrt{2}} k \right)$$

$$= \left(\frac{\sin t}{\sqrt{2}}, \frac{-\cos t}{\sqrt{2}}, 1/\sqrt{2} \right)$$

So we can use $\sqrt{2}B(t) = (\sin t, -\cos t, 1)$ as the normal vector for the plane. So, $\sqrt{2}B(\pi/2) = (1, 0, 1)$. So, eq of the plane is

$$x + z - \pi/2 \qquad = (1,0,1) \cdot (x-0, y-1, z-\pi/2) = 0$$

or equivalently $\boxed{x + z = \pi/2}$ ∎

Problem 3

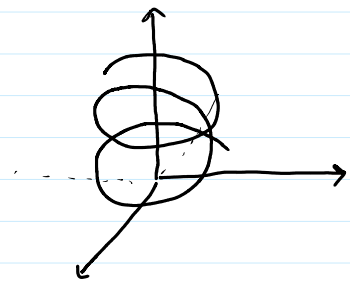
Wednesday, April 22, 2020 4:18 AM

Def Let $r(t)$ be a space curve. The <u>curvature</u> of $r(t)$ is given by:

$$K(t) = \frac{\|T'(t)\|}{\|r'(t)\|} \overset{(*)}{=} \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

Note: The equality $(*)$ needs proof, but it is beyond the scope of this exercise.

③ Show that the curvature of the helix $r(t) = (\cos t, \sin t, t)$ is constant.



Solution We compute $r'(t) \times r''(t)$:

$r'(t) = (-\sin t, \cos t, 1)$
$r''(t) = (-\cos t, -\sin t, 0)$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= -\cos t \begin{vmatrix} j & k \\ \cos t & 1 \end{vmatrix} + \sin t \begin{vmatrix} i & k \\ -\sin t & 1 \end{vmatrix}$$

$$= -\cos t\,(j - \cos t\, k) + \sin t\,(i + \sin t\, k)$$

$$= (\sin t, -\cos t, 1)$$

So,

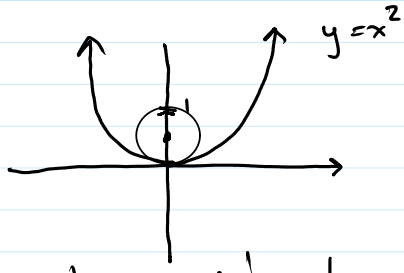
$$K(t) = \frac{\|r' \times r''\|}{\|r'\|^3} = \frac{\sqrt{\sin^2 t + \cos^2 t + 1}}{\sqrt{\sin^2 t + \cos^2 + 1}^{\,3}}$$

$$= \frac{\sqrt{2}}{\sqrt{2}^3} = \frac{1}{\sqrt{2}^2} = \frac{1}{2}. \quad \blacksquare$$

Fact The "best approximating circle" is the circle w/ radius $1/K(t)$

<u>Fact</u> The "best approximating circle" is the circle w/ radius $1/\kappa(t)$ that is contained in the osculating plane and is tangent to $r(t)$.

<u>osculating circle</u>



You can show that

$\kappa(0) = 2$

So osculating circle at the origin has radius $r = 1/\kappa(0) = 1/2$

Problem 4

Wednesday, April 22, 2020 7:53 AM

(4) Recall $B(t) = T(t) \times N(t)$

a) Show that $B'(t) \perp B(t)$

b) Show that $B'(t) \perp T(t)$

<u>Proof</u> (a) Recall $\|B(t)\| = 1$ so by Problem (1) we have $B'(t) \perp B(t)$.

(b) We need to show that $B'(t) \cdot T(t) = 0$.

We have:

$$
\begin{aligned}
B'(t) \cdot T(t) &= (T(t) \times N(t))' \cdot T(t) \\
&= (T'(t) \times N(t) + T(t) \times N'(t)) \cdot T(t) \\
&= \underbrace{(T'(t) \times N(t)) \cdot T(t)}_{0} + (T(t) \times N'(t)) \cdot T(t) \\
&= 0 + 0
\end{aligned}
$$

Note that $T'(t) \times N(t) = T'(t) \times \dfrac{T'(t)}{\|T'(t)\|} =$ zero vector so

$(T'(t) \times N(t)) \cdot T(t) = \vec{0} \cdot T(t) = 0.$ ∎

By (a) and (b) we know that $B' \perp B$ and $B' \perp T$. This means that B' is parallel to $B \times T$. So there exists a number $\tau(t) \in \mathbb{R}$ (depends on parameter t) such that

$$
\begin{aligned}
B' &= -\tau(t)\, B \times T \\
&= -\tau(t)\, N(t)
\end{aligned}
$$

The number $\tau(t)$ is called the Torsion and it measures the degree of twisting as we traverse $r(t)$.