Problem 1

Tuesday, May 19, 2020 10:40 PM

$\boxed{13.5}$

① You are walking on the graph of $f(x,y) = y\cos\pi x - x\cos\pi y + 10$ starting at the point $(2,1,13)$. Which direction should you walk to maintain a constant elevation (at least in a neighborhood of $(2,1)$)?

Solution

Let $v = (x,y)$ be a unit vector. We want to find $x, y \in \mathbb{R}$ such that

$$\nabla f(2,1) \cdot v = 0.$$

We have $\nabla f(x,y) = (-\pi y \sin\pi x - \cos\pi y,\ \cos\pi x + \pi x \sin\pi y)$

so $\nabla f(2,1) = (1, 1)$.

So we have the equation $0 = \nabla f(2,1) \cdot (x,y) = x+y$. But also v is a unit vector so we must have $x^2+y^2 = \|v\|^2 = 1$.

By substitution, $y = -x \Rightarrow x^2 + (-x)^2 = 1$
$\Rightarrow 2x^2 = 1$
$\Rightarrow x = \pm\frac{\sqrt{2}}{2}$
$\Rightarrow y = \pm\frac{\sqrt{2}}{2}$

So we can walk in the direction of $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$ ∎

14.1

② (a) Show that $f(x,t) = \sin(x-ct)$ satisfies the one-dimensional wave equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}. \iff f_{xx} = \frac{1}{c^2} f_{tt}$$

(b) Let $w = f(x,y)$ be a C^2 function of two variables and let $x = u+v$, $y = u-v$. Show that

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}. \iff w_{uv} = w_{xx} - w_{yy}$$

Proof of (a) We have

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\sin(x-ct)\right) \\
&= \frac{\partial}{\partial x}\cos(x-ct) \\
&= -\sin(x-ct) \\
&= -\frac{c^2}{c^2}\sin(x-ct) \\
&= -\frac{c}{c^2}\left(\frac{\partial f}{\partial t}\cos(x-ct)\right) \\
&= \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2}\sin(x-ct)\right) \\
&= \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}
\end{aligned}$$

∎

(b) Let $w = f(x,y)$ be a function of two variables and let $\underbrace{x = u+v, y = u-v}_{\mathbb{R}^2 \to \mathbb{R}^2}$. Show that

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}. \iff w_{uv} = w_{xx} - w_{yy} \quad (u,v) \longmapsto (u+v, u-v)$$

Suppose we want to compute $\frac{\partial w}{\partial v}$ using chain rule $\left(f: \mathbb{R}^m \to \mathbb{R}^n,\ g: \mathbb{R}^n \to \mathbb{R}^p \quad \underbrace{D(g\circ f)(x)}_{p\times m} = \underbrace{Dg(f(x))}_{p\times n} \cdot \underbrace{Df(x)}_{n\times m} \right)$

$$\begin{bmatrix} \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix}}^{1\times 2} \cdot \overbrace{\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}}^{2\times 2}$$

$$= \begin{bmatrix} \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} \end{bmatrix}$$

Compare entries: $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} \quad (*)$

Compare entries : $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$ $(*)$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}$$

Now to prove the claim....

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}\right) \qquad \left(\begin{matrix} x = u+v \\ y = u-v \end{matrix}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x}\right) - \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial y}\right) \qquad \left(\text{Let } h_1 = \frac{\partial w}{\partial x} \quad h_2 = \frac{\partial w}{\partial y}\right)$$

$$\overset{(*)}{=} \frac{\partial h_1}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial h_1}{\partial y}\frac{\partial y}{\partial u} - \left(\frac{\partial h_2}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial h_2}{\partial y}\frac{\partial y}{\partial u}\right)$$

$$= \frac{\partial h_1}{\partial x} + \frac{\partial h_1}{\partial y} - \frac{\partial h_2}{\partial x} - \frac{\partial h_2}{\partial y}$$

$$= \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y \partial x} - \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2}$$

$$= \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \qquad \left(\begin{matrix}\text{By Clairaut's Thm} \\ \frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}\end{matrix}\right)$$

$\blacksquare$