Problem 1

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① Find $\frac{\partial}{\partial s} f \circ T(s,t)$ where $f(u,v) = \cos u \sin v$ and $T(s,t) = (t^2, s^2)$.

<u>Thm</u> (Chain Rule) Let $g: \mathbb{R}^n \to \mathbb{R}^m$ and $f: \mathbb{R}^m \to \mathbb{R}^p$ be functions such that $f \circ g$ is defined. Suppose g is differentiable at $x_0 \in \mathbb{R}^n$ and f is differentiable at $y_0 = g(x_0) \in \mathbb{R}^m$. Then $f \circ g$ is differentiable at x_0 and

matrix multiplication

$$\underbrace{D(f \circ g)(x_0)}_{p \times n \text{ matrix}} = \underbrace{Df(y_0)}_{p \times m} \cdot \underbrace{Dg(x_0)}_{m \times n}$$

① Find $\frac{\partial}{\partial s} f \circ T(s,t)$ where $f(u,v) = \cos u \sin v$ and $T(s,t) = (t^2, s^2)$.

<u>Solution</u> Find Df and DT:

$$Df = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix} = \begin{bmatrix} -\sin u \sin v & \cos u \cos v \end{bmatrix}$$

$$DT = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & 2t \\ 2s & 0 \end{bmatrix}$$

So

$$\begin{bmatrix} \frac{\partial f \circ T}{\partial s} & \frac{\partial f \circ T}{\partial t} \end{bmatrix} = D(f \circ T) = \begin{bmatrix} -\sin u \sin v & \cos u \cos v \end{bmatrix} \begin{bmatrix} 0 & 2t \\ 2s & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2s \cos u \cos v & -2t \sin u \sin v \end{bmatrix}$$

$$= \begin{bmatrix} 2s \cos t^2 \cos s^2 & -2t \sin t^2 \sin s^2 \end{bmatrix}$$

Thus, $\frac{\partial f \circ T}{\partial s} = 2s \cos t^2 \cos s^2$.

② Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable. Find formulas for $\frac{\partial f}{\partial \rho}, \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi}$ in terms of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ where (ρ, θ, ϕ) are spherical coordinates.

Solution Substitute

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

(This is a map $g : \mathbb{R}^3 \to \mathbb{R}^3$
$(\rho, \theta, \phi) \longmapsto (x(\rho,\theta,\phi), y(\rho,\theta,\phi), z(\rho,\theta,\phi))$)

By def $Df = \left[\frac{\partial f}{\partial \rho} \;\; \frac{\partial f}{\partial \theta} \;\; \frac{\partial f}{\partial \phi}\right]$ and by chain rule

$$D(f \circ g) = \underline{Df(g)} \circ \underline{Dg}$$
$$= \left[\frac{\partial f}{\partial x} \;\; \frac{\partial f}{\partial y} \;\; \frac{\partial f}{\partial z}\right] \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{bmatrix}$$

Comparing entries gives:

$$\frac{\partial f}{\partial \rho} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \rho} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial \rho}$$
$$= \sin\phi \cos\theta \frac{\partial f}{\partial x} + \sin\phi \sin\theta \frac{\partial f}{\partial y} + \cos\phi \frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial \theta}$$
$$= -\rho \sin\phi \sin\theta \frac{\partial f}{\partial x} + \rho \sin\phi \cos\theta \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial \phi}$$
$$= \rho \cos\phi \cos\theta \frac{\partial f}{\partial x} + \rho \cos\phi \sin\theta \frac{\partial f}{\partial y} - \rho \sin\phi \frac{\partial f}{\partial z}$$

$\blacksquare$

Problem 3

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(3) Define $y(x)$ implicitly via $G(x, y(x)) = K$ where $G: \mathbb{R}^2 \to \mathbb{R}$.
Prove the implicit differentiation formula: if $y(x)$ and G are differentiable and $\frac{\partial G}{\partial y} \neq 0$, then

$$\frac{dy}{dx} = -\frac{\partial G/\partial x}{\partial G/\partial y}.$$

<u>Proof</u> Define $H(x) = G(x, y(x))$. Note $H: \mathbb{R} \to \mathbb{R}$ and $H = G \circ F$ where $F(x) = (x, y(x))$. By the chain rule

$$DH = DG \cdot DF$$

$$= \begin{bmatrix} \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{dy}{dx} \end{bmatrix}$$

(Note $y: \mathbb{R} \to \mathbb{R}$ so $\frac{\partial y}{\partial x} = \frac{dy}{dx}$)

$$= \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} \cdot \frac{dy}{dx}$$

But also $H(x) = K$ so $DH = 0$. So

$$0 = DH = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} \cdot \frac{dy}{dx}$$

Since $\frac{\partial G}{\partial y} \neq 0$, we can divide by $\frac{\partial G}{\partial y}$ to obtain:

$$\frac{dy}{dx} = -\frac{\partial G/\partial x}{\partial G/\partial y}$$

$\blacksquare$

<u>Ex</u> Find $\frac{dy}{dx}$ where y is defined implicitly by $x^2 + y^2 = R^2$

By formula: $\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$

By Calc I: $2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{\partial y}{\partial x} = -\frac{2x}{2y} = -\frac{x}{y}$.

Problem 4

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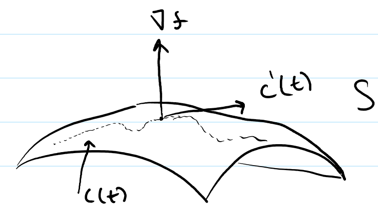
continuously differentiable

④ Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a C^1 map and suppose (x_0, y_0, z_0) lies on the level surface S defined by $f(x,y,z) = K$. Show that $\nabla f(x_0,y_0,z_0)$ is normal to S.



Proof Let $c: \mathbb{R} \to \mathbb{R}^3$, $c(t) = (x(t), y(t), z(t))$ be any curve lying in S such that $c(0) = (x_0, y_0, z_0)$.

Goal: Show that $\nabla f(x_0,y_0,z_0) \cdot c'(0) = 0$.

We have $\nabla f \cdot c' = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$

$$= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

$$= \frac{\partial}{\partial t}(f \circ c)$$

$$= \frac{d}{dt}(f \circ c)$$

So we have the relation $\nabla f \cdot c' = \frac{d}{dt}(f \circ c)$. Evaluate at $t=0$

$$\Rightarrow \quad \nabla f(x_0,y_0,z_0)\, c'(0) = \frac{d}{dt}(f \circ c)(0) = \frac{d}{dt} K = 0. \quad \blacksquare$$

So the plane tangent to S a (x_0, y_0, z_0) is given by

$$\nabla f(x_0,y_0,z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0 \qquad (*)$$

If G is the graph of $f(x,y)$ and $h(x,y,z) = f(x,y) - z$, then the level curve $h(x,y,z) = 0$ is the graph G. So by $(*)$ the tangent plane is (since $\nabla h = (f_x, f_y, -1)$)

$$f_x(x_0,y_0)(x - x_0) + f_y(y - y_0) - (z - z_0) = 0$$

Cool Fact In general, let $f: \mathbb{R}^n \to \mathbb{R}$, you can define a tangent hyperplane (of dimension $\mathbb{R}^m$) via

to the graph of f

$$\nabla f(\vec{x}_0)(\vec{x}-\vec{x}_0)=0.$$