① Compute the following limits, if they exist. Justify your computations.

a) $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2+2}$$

b) $$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$$

c) $$\lim_{(x,y)\to(0,0)} (3x^2+3y^2)\ln(x^2+y^2)$$

d) $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Solution

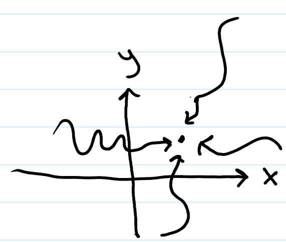
a) $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2+2} = 0$$ since $f(x,y) = \dfrac{xy}{x^2+y^2+2}$ is continuous at $(0,0)$. ∎

b) $$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$$

This function is not continuous at $(0,0)$. To show that the limit does not exist, we need to find two different paths that approach $(0,0)$ that give different values in the limit.

For instance, along the path where $x=0$, we have

$$\lim_{(0,y)\to(0,0)} \frac{(-y)^2}{y^2} = 1$$



But along the path $y=x$ we have

$$\lim_{(x,x)\to(0,0)} \frac{(x-x)^2}{x^2+x^2} = 0.$$ So the limit doesn't exist! ∎

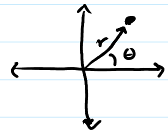
c) $$\lim_{(x,y)\to(0,0)} (3x^2+3y^2)\ln(x^2+y^2)$$

This function is not continuous at $(0,0)$ since $\ln(0)$ is not defined. We can change the limit to polar coordinates by setting $r = \sqrt{x^2+y^2}$. Notice that

$$\lim \sqrt{x^2+y^2} = 0$$

We can change the limit to polar coordinates by setting $r = \sqrt{x^2+y^2}$. Notice that

$$\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$$



So,

$$\lim_{(x,y)\to(0,0)} (3x^2+3y^2)\ln(x^2+y^2) = \lim_{r\to 0} 3r^2 \ln r^2$$

$$= 3 \cdot \lim_{r\to 0} \frac{\ln r^2}{1/r^2}$$

$$\overset{L'H}{=} 3 \lim_{r\to 0} \frac{\frac{1}{r^2}\cdot 2r}{-2/r^3}$$

$$= 3 \lim_{r\to 0} \frac{1/r}{-1/r^3}$$

$$= 3 \lim_{r\to 0} -r^2 = 3\cdot 0$$

$$= 0$$

d) $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

This function is not continuous at $(0,0)$. Along the path $y=0$ we have

$$\lim_{(x,0)\to(0,0)} \frac{x\cdot 0}{x^2+0^2} = 0.$$

Along the path $y=x$ we have

$$\lim_{(x,x)\to(0,0)} \frac{x^2}{x^2+x^2} = \frac{1}{2}.$$

Thus, the limit doesn't exist.

Problem 2

Tuesday, April 28, 2020 9:19 PM

(2) Suppose $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are functions such that

$$f(x,y) = g(xy) \qquad (\text{for all } (x,y) \in \mathbb{R}^2)$$

If $(a,b) \in \mathbb{R}^2$ and g is continuous at ab, then $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and is equal to $\lim_{t \to ab} g(t) = g(ab)$.

<u>Proof</u> Define a function $h: \mathbb{R}^2 \longrightarrow \mathbb{R}$ via $h(x,y) = x \cdot y$. Then

$$\begin{aligned}
\lim_{(x,y)\to(a,b)} f(x,y) &= \lim_{(x,y)\to(a,b)} g(xy) \\
&= \lim_{(x,y)\to(a,b)} g(h(x,y)) \\
&= g\Big(\lim_{(x,y)\to(a,b)} h(x,y)\Big) && [\text{since } g \text{ is continuous at } ab] \\
&= g(ab) && [\text{since } h \text{ is continuous at } (a,b)] \\
&= \lim_{t \to ab} g(t).
\end{aligned}$$

∎

Problem 3

Tuesday, April 28, 2020 9:58 PM

③ Compute the following limits, if they exist. Justify all computations

a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{y}$

b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\cos(xy)-1}{x^2y^2}$

c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

Solution

looks like $\dfrac{e^t-1}{t}$

a) $$\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{y} = \lim_{(x,y)\to(0,0)} x\cdot\frac{e^{xy}-1}{xy}$$

If we know that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{xy}$ exists, then we can write

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{y} = \left(\lim_{(x,y)\to(0,0)} x\right)\cdot\left(\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{xy}\right)$$
$$= 0\cdot 1$$
$$= 0.$$

To compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{xy}$, consider

$$g(t) = \begin{cases} \dfrac{e^t-1}{t}, & t\neq 0 \\ 1, & t=0 \end{cases}$$

which is continuous at 0. Notice that $g(xy) = \dfrac{e^{xy}-1}{xy}$. So,

by problem ②,

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{xy} = \lim_{t\to 0} g(t)$$
$$= 1.$$

b) $\lim_{(x,y)\to(0,0)} \frac{\cos(xy)-1}{x^2y^2}$

Consider the continuous function $g(t) = \begin{cases} \frac{\cos t - 1}{t^2}, & t \neq 0 \\ -\frac{1}{2}, & t = 0 \end{cases}$.

Note $g(xy) = \frac{\cos(xy)-1}{x^2y^2}$ for all $(x,y) \in \mathbb{R}^2$. Thus,

$$\lim_{(x,y)\to(0,0)} \frac{\cos(xy)-1}{x^2y^2} = \lim_{t\to 0} g(t)$$

$$= -\frac{1}{2}. \qquad \blacksquare$$

c) $$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{\sin r}{r}$$

$$= 1 \qquad \blacksquare$$