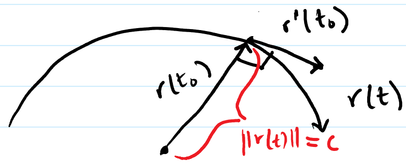
Problem 1

Monday, April 20, 2020 4:58 AM

① Let $r(t)$ be a parameterization of a curve. Suppose that $\|r(t)\| = c$ for all $t \in \mathbb{R}$. Show that $r(t)$ and $r'(t)$ are orthogonal for all $t \in \mathbb{R}$.

The Picture :



Proof We need to show that $r(t) \cdot r'(t) = 0$. By assumption, we have $c^2 = \|r(t)\|^2$

Thus,
$$0 = \frac{d}{dt} c^2 = \frac{d}{dt} \|r(t)\|^2 = \frac{d}{dt}(r(t) \cdot r(t))$$
$$\overset{(*)}{=} r'(t) \cdot r(t) + r(t) \cdot r'(t)$$
$$= 2 r(t) r'(t).$$

(*) Product rule for dot product

Dividing by 2 yields $r(t) \cdot r'(t) = 0$. □

Def Given a curve $r(t)$ we can define the unit tangent vector:

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

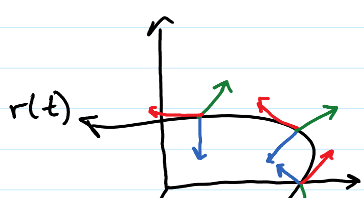
By definition $\|T(t)\| = 1$ for all $t \in \mathbb{R}$. By Problem 1, we know that $T(t)$ is orthogonal to $T'(t)$. So, we define the Unit Normal Vector

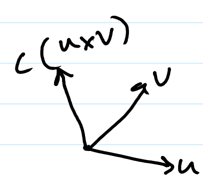
$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

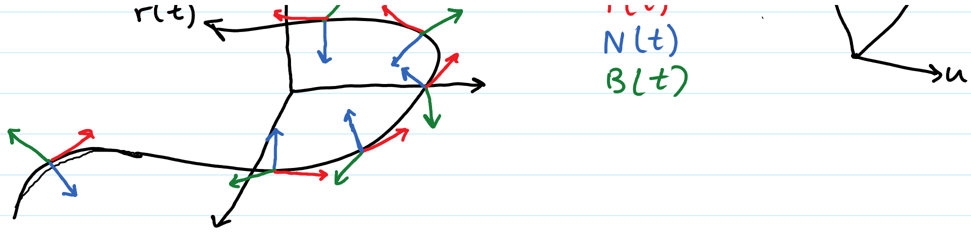
Further, we define the Unit Binormal Vector by

$$B(t) = T(t) \times N(t)$$

$B(t)$ is a unit vector since T and N are unit vectors.







Note that $\{T(t), N(t), B(t)\}$ at each point $t \in \mathbb{R}$ can be obtained from the vectors $\{i, j, k\}$ via translation and rotation. In linear algebra this is called an orthonormal basis.

The "TNB Frame" is used to describe the geometry of a space curve, such as orientation, curvature, and torsion.

Def Let $r(t)$ be a parameterized space curve and P a point on the curve.

- The Normal plane at P is the plane containing P, N, & B.
- The Osculating Plane at P is the plane containing P, T, & N.

Recall
For a parameterized curve $r(t)$ we have

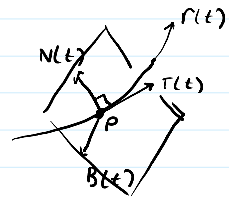
$T(t) = \dfrac{r'(t)}{\|r'(t)\|}$ — unit tangent vector

$N(t) = \dfrac{T'(t)}{\|T'(t)\|}$ — unit normal vector

$B(t) = T(t) \times N(t)$ — unit binormal vector

(2) The helix is given by $r(t) = (\cos t, \sin t, t)$. When $t = \pi/2$, find:

(a) The Normal plane
(b) The Osculating plane.



Solution:

(a) By definition, $r'(t)$ is orthogonal to both $N(t)$ and $B(t)$. When $t = \pi/2$,

we have $r(\pi/2) = (\cos \pi/2, \sin \pi/2, \pi/2) = (0, 1, \pi/2)$ and $r'(\pi/2) = (-\sin \pi/2, \cos \pi/2, 1)$
$= (-1, 0, 1)$

So eq. of Normal Plane is:

$-x + z - \pi/2 \qquad = (-1, 0, 1) \cdot (x - 0, y - 1, z - \pi/2) = 0$

or equivalently $\quad x - z = -\pi/2$.

(b) By definition, the vector $B(t)$ is orthogonal to both $T(t)$ and $N(t)$.

We have: $T(t) = \dfrac{r'(t)}{\|r'(t)\|} = \dfrac{(-\sin t, \cos t, 1)}{\sqrt{2}}$

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{\frac{1}{\sqrt{2}}(-\cos t, -\sin t, 0)}{\frac{1}{\sqrt{2}} \cdot \sqrt{1}} = (-\cos t, -\sin t, 0)$$

$$B(t) = T(t) \times N(t) = \begin{vmatrix} i & j & k \\ -\sin t & \cos t & 1 \\ \overset{+}{-\cos t} & \overset{-}{-\sin t} & \overset{+}{0} \end{vmatrix}$$

$$= -\cos t \begin{vmatrix} j & k \\ \cos t & 1 \end{vmatrix} + \sin t \begin{vmatrix} i & k \\ -\sin t & 1 \end{vmatrix}$$

$$= -\cos t (j - \cos t\, k) + \sin t (i + \sin t\, k)$$

$$= (\sin t, -\cos t, 1)$$

So the normal vector is $B(\pi/2) = (\sin\pi/2, -\cos\pi/2, 1)$

$$= (1, 0, 1)$$

So eq of osculating plane is

$x + z - \pi/2$ $\qquad (1,0,1)\cdot(x-0, y-1, z-\pi/2) = 0$

or equivalently $\quad x + z = \pi/2$.

Def Let $r(t)$ be a space curve. The curvature of $r(t)$ is given by:

$$K(t) = \frac{\|T'(t)\|}{\|r'(t)\|} \overset{(*)}{=} \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

Note: The equality $(*)$ needs proof, but it is beyond the scope of this exercise.

③ Show that the curvature of the helix $r(t) = (\cos t, \sin t, t)$ is constant.

Solution: We compute:

$$r'(t) = (-\sin t, \cos t, 1)$$
$$r''(t) = (-\cos t, -\sin t, 0)$$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= -\cos t \begin{vmatrix} j & k \\ \cos t & 1 \end{vmatrix} + \sin t \begin{vmatrix} i & k \\ -\sin t & 1 \end{vmatrix}$$

$$= -\cos t\,(j - \cos t\, k) + \sin t\,(i + \sin t\, k)$$
$$= (\sin t, -\cos t, 1)$$

So, $$K(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = \frac{\sqrt{\sin^2 t + \cos^2 t + 1}}{\sqrt{\sin^2 t + \cos^2 t + 1}^{\,3}} = \frac{\sqrt{2}}{\sqrt{2}^{\,3}} = \frac{1}{2}.$$ $\square$

(4) Recall $B(t) = T(t) \times N(t)$

a) show that $B'(t) \perp B(t)$
b) show that $B'(t) \perp T(t)$

Proof (a) By Problem 1, since $\|B(t)\| = 1$ for all $t \in \mathbb{R}$, we have $B(t) \cdot B'(t) = 0$.

(b) we have

$$\begin{aligned} B'(t) \cdot T(t) &= (T(t) \times N(t))' \cdot T(t) \\ &= \left(T'(t) \times N(t) + T(t) \times N'(t)\right) \cdot T(t) \\ &= \left(T'(t) \times N(t)\right) \cdot T(t) + \left(T(t) \times N'(t)\right) \cdot T(t) \\ &= 0 + 0 \end{aligned}$$

Note that $\left(T'(t) \times N(t)\right) \cdot T(t) = 0$ since $T'(t) \times N(t) = T'(t) \times \frac{T'(t)}{\|T'(t)\|} = (0,0,0)$.
Also, $(T(t) \times N'(t)) \cdot T(t) = 0$ since $T(t) \times N'(t)$ is $\perp$ to $T(t)$.

$\blacksquare$

By part (a) and (b), we deduce that there is a constant $\tau(t) \in \mathbb{R}$ such that

$$\begin{aligned} B'(t) &= \tau(t)\, (T(t) \times B(t)) \\ &= -\tau(t)\, N(t) \end{aligned}$$

The constant $\tau(t)$ is called the Torsion, and it describes the degree of twisting that occurs as we traverse $r(t)$.