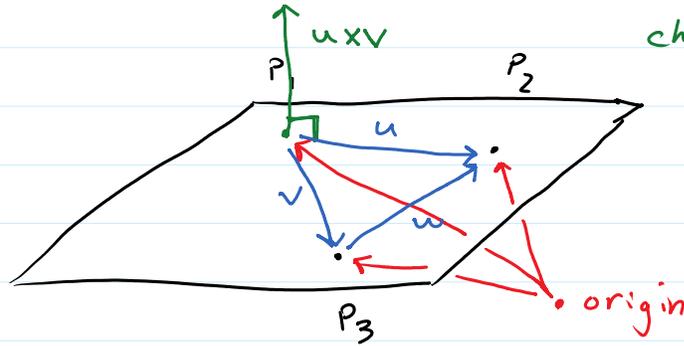


Problem 1

Sunday, April 5, 2020 10:43 PM

① Show that the points $P_1 = (1, -1, 1)$, $P_2 = (0, 1, -2)$, and $P_3 = (-2, 1, 0)$ are coplanar, i.e., P_1, P_2, P_3 lie in the same plane.



check: $w \cdot (u \times v) = 0$.

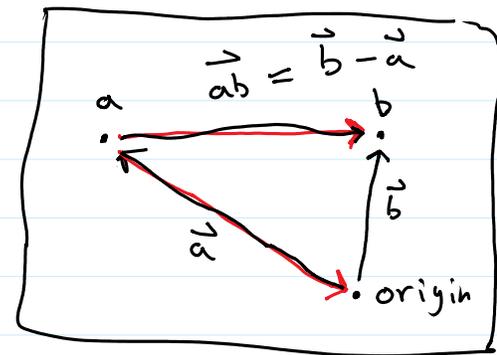
Find a strategy to prove the points lie in the same plane!

P_1, P_2, P_3 are coplanar if and only if the vectors u, v, w are coplanar.

Compute u, v, w :

$$u = P_2 - P_1 = (-1, 2, -3), v = P_3 - P_1 = (-3, 2, -1)$$

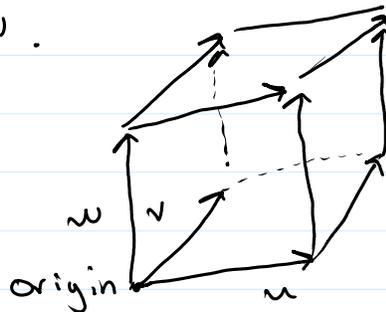
$$w = P_2 - P_3 = (2, 0, -2)$$



We see that u, v, w are coplanar if and only if $w \cdot (u \times v) = 0$. We have

$$\begin{aligned} \boxed{w \cdot (u \times v)} &= (2, 0, -2) \begin{vmatrix} i & j & k \\ -1 & 2 & -3 \\ -3 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & -2 \\ -1 & 2 & -3 \\ -3 & 2 & -1 \end{vmatrix} \\ \text{Scalar triple product} & \\ &= (2, 0, -2) \left(\begin{vmatrix} 2 & -3 \\ 2 & -1 \end{vmatrix} i - \begin{vmatrix} -1 & -3 \\ -3 & -1 \end{vmatrix} j + \begin{vmatrix} -1 & 2 \\ -3 & 2 \end{vmatrix} k \right) \\ &= 2 \begin{vmatrix} 2 & -3 \\ 2 & -1 \end{vmatrix} + 0 \begin{vmatrix} -1 & -3 \\ -3 & -1 \end{vmatrix} - 2 \begin{vmatrix} -1 & 2 \\ -3 & 2 \end{vmatrix} \\ &= 0. \end{aligned}$$

Cool Fact: $w \cdot (u \times v)$ = Volume of the parallelepiped spanned by w, u, v .



Problem 2

Sunday, April 5, 2020 11:16 PM

② Consider the lines given by

$$r_1(t) = (1, 1, 0) + t(1, -1, 2), \quad -\infty < t < \infty$$

$$r_2(s) = (2, 0, 2) + s(-1, 1, 0), \quad -\infty < s < \infty.$$

a) Show that r_1 and r_2 intersect.

b) Find an equation of the plane that contains both r_1 and r_2 .

Solution (a) $r_1(t) = (1+t, 1-t, 2t)$

$$r_2(s) = (2-s, s, 2)$$

Set them equal: $(1+t, 1-t, 2t) = (2-s, s, 2)$

So $2t=2 \Rightarrow t=1$. Setting $t=1$ we obtain

$$(2, 0, 2) = (2-s, s, 2)$$

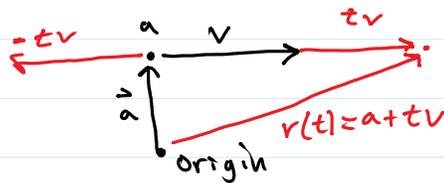
So $s=0$. So the lines intersect at the point $(2, 0, 2)$! \square

(b) To find an equation of a plane, we need a vector perpendicular to the plane and a point in the plane. We already know $p=(2, 0, 2)$ is in the plane.

Note A parameterization of a line looks like

$$r(t) = a + tv$$

where a is a point in the line and v points in the direction of the line.



we need to take the cross product of $v_1 = (1, -1, 2)$, $v_2 = (-1, 1, 0)$

$$v_1 \times v_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} i - \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} j + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} k$$

$$= -2i - 2j = (-2, -2, 0)$$

So the plane containing r_1, r_2 is given by

E_1 of a plane!

So the plane containing r_1, r_2 is given by

$$(-2, -2, 0) \cdot (x-2, y-0, z-2) = 0$$

\vec{n} of a plane!
 $(\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0)$

\Rightarrow

$$-2x + 4 - 2y = 0$$

\Rightarrow

$$\boxed{-2x - 2y = -4}$$



Problem 3

Tuesday, April 14, 2020 7:46 PM

③

a) Let $v, w \in \mathbb{R}^n$. If $\|v\| = \|w\|$, prove that $v+w$ and $v-w$ are orthogonal.

b) Suppose that three points a, b, c lie on a circle such that a and b are antipodal. Use part (a) to show that $\triangle abc$ is a right triangle.

a) Show that $(v+w) \cdot (v-w) = 0$.

Proof Suppose $\|v\| = \|w\|$. Then

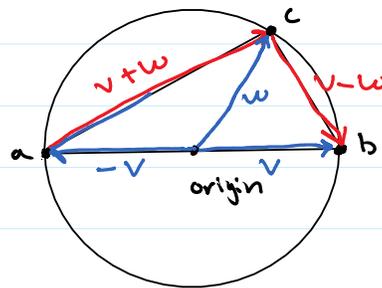
$$\begin{aligned} (v+w) \cdot (v-w) &= v \cdot v - v \cdot w + w \cdot v - w \cdot w \\ &= v \cdot v - w \cdot w \\ &= \|v\|^2 - \|w\|^2 = \|v\|^2 - \|v\|^2 \\ &= 0. \end{aligned}$$

Note

$$\|u\| = \sqrt{u \cdot u}$$

□

Picture for (b):



Prove that $\triangle abc$ is a right triangle!

$\triangle abc$ is a right triangle if and only if $v+w$ is orthogonal to $v-w$. But we know $\|w\| = \|v\|$ since $\|w\| = r = \|v\|$ where r is the radius of the circle. By (a) $v+w, v-w$ are perpendicular. □

Problem 4

Tuesday, April 14, 2020 8:07 PM

(4) Let $u, v, w \in \mathbb{R}^3$. Suppose that there exists $a, b \in \mathbb{R}$ such that $w = au + bv$. Find the value of $u \cdot (v \times w)$.

Solution

$$\begin{aligned} u \cdot (v \times w) &= u \cdot (v \times (au + bv)) \\ &= u \cdot (v \times au + v \times bv) \\ &= u \cdot (a(v \times u) + b(v \times v)) \\ &= au \cdot \underline{(v \times u)} + bu \cdot \underline{(v \times v)} \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

Note that $v \times u$ is orthogonal to au so $au \cdot (v \times u) = 0$.
Also, $v \times v = (0, 0, 0)$ so $bu \cdot (v \times v) = 0$. \rightarrow (see lemma) \square

Lemma For all $v \in \mathbb{R}^3$, $v \times v = 0$.

Proof Suppose $v = (x, y, z)$. Then

$$\begin{aligned} v \times v &= \begin{vmatrix} i & j & k \\ x & y & z \\ x & y & z \end{vmatrix} \\ &= |yz| i - |xz| j + |xy| k \\ &= 0i - 0j + 0k = 0. \end{aligned}$$

\square