Inverse anisotropic conductivity from power densities in two dimensions

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Sept. 3, 2014
Pontificia Universidad Católica de Chile

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1. Introduction

2. A coupled-physics approach to inverse conductivity

3. Inverse conductivity from power densities - resolution

4. Numerical simulations
Overview

**Inverse problem:** Find $x \in \mathcal{X}$ such that $y = M(x)$, given data $y \in \mathcal{Y}$ and model $M$.

**Examples:**

**X-Ray CT:** a function $\gamma \leftarrow$ its line integrals. $M$: X-Ray transform.

**Calderón’s problem (EIT):** $\gamma \leftarrow \Lambda_\gamma$, $M$: conductivity eq.

**Seismography:** sound speed in Earth’s crust $\leftarrow$ wave traveltimes.

**Inverse wave problem:** $u|_{t=0} \leftarrow u|_{\partial \Omega \times [0, T]}$, $M$: wave eq.
Introduction

Analysis of inverse problems

The theoretical analysis of an inverse problem consists in

- **Uniqueness** assessment: \( M(x_1) = M(x_2) \implies x_1 = x_2 \)
- **Stability** assessment (rules resolution in practice)
  Hilbert scale of stability:
  - well-posed problems (\( \delta M \) small \( \implies \delta x \) small)
  - mildly ill-posed problems (\( \delta M \) small \( \implies \delta x \) not too large)
  - severely ill-posed (\( \delta M \) small \( \implies \delta x \) too large)

- Derivation of **reconstruction formulas**
- Numerical implementation/simulation

▶ Applications to **imaging sciences**
  (medical, geophysical, atmospheric, etc...).
Two (linear) examples

1. $M_1[f](x) = \int_0^x f(t) \, dt$.
2. $M_2[f](x) = u(x, T)$ ($T > 0$ fixed), where $u$ solves
   \[
   \partial_t u = \partial_{xx} u \quad (t > 0, x \in \mathbb{R}), \quad u(x, 0) = f(x).
   \]

Both problems are injective. In a noiseless world, $f(x) = \frac{d}{dx} M_1[f](x)$, and in particular,
\[
\|f - g\|_{L^2(\mathbb{R})} \leq \|M_1[f] - M_1[f]\|_{H^1(\mathbb{R})}.
\]

For the second problem, $f(x) = \mathcal{F}_{\xi \rightarrow x}^{-1}(e^{T \xi^2} \mathcal{F}_{x \rightarrow \xi} M_2[f])$, though there is no $(s, p, C)$ such that
\[
\|f - g\|_{H^s(\mathbb{R})} \leq C\|M_2[f] - M_2[g]\|_{H^{s+p}(\mathbb{R})}.
\]

Problem 1 is mildly ill-posed while problem 2 is severely ill-posed.

▶ Stability quantifies the resolution available on reconstructions.
Motivation

Underlying goal: To **improve resolution** in soft tissue medical imaging modalities (e.g. from centimetric to millimetric).

- **Mechanical, optical** and **electrical** properties of tissues display **good contrast** for e.g. tumor detection, lung activity monitoring.

- Their corresponding imaging modalities are very **poorly resolved**.

- Inverse problems: mathematically **severely ill-posed**.

**Figure**: Conductivity map of human chest

[Kerrouche et al. ’01]
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The inverse conductivity problem

The model: $X \subset \mathbb{R}^n$ bounded domain.

- **Calderón’s problem:**
  Does $\Lambda_\gamma$ determine $\gamma$ uniquely? stably?
  
  [Calderón ’80]

- **Power density problem:**
  Does $\mathcal{H}_\gamma$ determine $\gamma$ uniquely? stably?

Note: $\int_X \mathcal{H}_\gamma u = \int_{\partial X} g \Lambda_\gamma g$

$\gamma$: uniformly elliptic conductivity tensor.
Derivation of power densities - 1/2

By ultrasound modulation

Physical focusing

[Ammann et al. ’08]

Synthetic focusing

[Kuchment-Kunyansky ’10]
[Bal-Bonnetier-M.-Triki ’11]

Small perturbation model:

\[
\frac{(M_\epsilon - M_0)}{\epsilon} \text{ gives an approximation of } \nabla u_0 \cdot \gamma \nabla u_0 \text{ at } x_0.
\]
Derivation of power densities - 2/2

By thermoelastic effects (Impedance-Acoustic CT)

1: voltage is prescribed at $\partial X$
   \[ u|_{\partial X} = g(x) \delta(t) \]

2: currents are generated inside the domain
   \[ \nabla \cdot (\gamma \nabla u) = 0 \]

3: the energy absorbed generates elastic waves
   \[ \frac{1}{v_s^2} \frac{\partial^2 \rho}{\partial t^2} - \Delta \rho = 0 \]
   \[ \rho|_{t=0} = \Gamma \mathcal{H}_\gamma[g] \]
   \[ \partial_t \rho|_{t=0} = 0 \]

4: waves are measured at $\partial X$ by ultrasound transducers

One reconstructs $\Gamma \mathcal{H}_\gamma = \Gamma \nabla u \cdot \gamma \nabla u$ over $X$ ($\Gamma$: Grüneisen coefficient)

[Gebauer-Scherzer ’09]
Theoretical comparison of inversions

From Dirichlet-to-Neumann map (latest 2D results):

- **uniqueness:** holds for isotropic $L^\infty$ [Astala-Päivärinta ’05]. Anisotropic case: lack of injectivity completely characterized [Astala-Lassas-Päivärinta ’05].

- **stability:** logarithmic, i.e. severely ill-posed. Poor resolution in practice. [Alessandrini ’88], [Santacesaria ’11]

From (”enough”) power density measurements:

Define $|\gamma| := \det \gamma$ and $\tilde{\gamma} := |\gamma|^{-\frac{1}{2}} \gamma$ ($\tilde{\gamma}$ satisfies $\det \tilde{\gamma} = 1$).

- **uniqueness:** both $|\gamma|^\frac{1}{2}$ and $\tilde{\gamma}$ are uniquely reconstructible via explicit, algebraic algorithms

- **stability:** well-posed (Lipschitz) in $W^{1,\infty}$ for $|\gamma|^\frac{1}{2}$, ill-posed of order 1 (Hölder) for $\tilde{\gamma}$. 
Power density measurements - References

References on resolution of the power density problem:

- 2D isotropic  [Capdeboscq et al. ’09].
- 2D-3D isotropic linearized  [Kuchment-Kunyansky ’11].
- 2D-3D isotropic  [Bal-Bonnetier-M.-Triki, IPI ’12].
- nD isotropic:  [M.-Bal, IPI ’12] [Kocyigit ’12]
- 2D anisotropic:  [M.-Bal, IP ’12] (today’s talk)
- nD anisotropic:  [M.-Bal, CPDE ’13].

Pseudodifferential analysis on the linearized problem:

- Isotropic case  [Kuchment-Steinhauer, ’12].
- nD anisotropic case  [Bal-Guo-M., IPI ’13].

Resolution from a single power density - isotropic case:

- Newton-based numerical methods to recover \((u, \gamma)\)
  [Ammari et al. ’08, Gebauer-Scherzer ’09].
- Theoretical work on the Cauchy problem  [Bal ’11].
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The problem

Problem: recover $\gamma = (|\gamma|^{\frac{1}{2}}, \tilde{\gamma})$ from the knowledge of
$H_{ij}(x) = \gamma \nabla u_i \cdot \nabla u_j$, where $u_i$ solves

$$\nabla \cdot (\gamma \nabla u_i) = 0, \quad u_i|_{\partial \Omega} = g_i, \quad 1 \leq i \leq 3.$$ 

"Reparameterize" the problem: Define $A = \gamma^{\frac{1}{2}}$ and $S_i := A \nabla u_i$, all
unknowns. The $S_i$’s satisfy

$$\nabla \cdot (AS_i) = 0, \quad (J \nabla) \cdot (A^{-1} S_i) = 0, \quad 1 \leq i \leq 3,$$

where $J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and the data takes the form $H_{ij} = S_i \cdot S_j$
(Grammian matrix).
Intermediate equations of interest

From two solutions \((u_1, u_2)\): write a decomposition of \(S = [S_1\vert S_2] = [A\nabla u_1\vert A\nabla u_2]\) of the form

\[ S = Q(\theta)R(H_{11}, H_{12}, H_{22}) \] (e.g. "QR" or \(S = Q(\theta)\left[ \begin{array}{cc} H_{11} & H_{12} \\ \text{sym} & H_{22} \end{array} \right]^{\frac{1}{2}}\)),

where \(Q(\theta) = \left[ \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right] \). Using the previous PDE’s, one derives:

\[ \nabla \log |\gamma|^{\frac{1}{2}} = N + (\nabla H_{ij} \cdot \hat{A}S_i)\hat{A}^{-1}S_j, \]

\[ \hat{A}^2 \nabla \theta + [\hat{A}_2, \hat{A}_1] = \hat{A}^2 V - \frac{1}{2} JN, \quad \hat{A} := \hat{\gamma}^{\frac{1}{2}}, \]

where \(V := \frac{1}{2} H_{11}^{-1} \nabla H_{12} \) and \(N := \nabla \log d\), with \(d := (H_{11}H_{22} - H_{12}^2)^{\frac{1}{2}}\).

Legend: known data, anisotropic structure, unknown.
Reconstruction of $\tilde{\gamma}$

Using the above equation for two systems $(u_1, u_2)$ and $(u_1, u_3)$:

$$\tilde{A}^2 X = J Y, \quad Y := \nabla \frac{\det(\nabla u_1, \nabla u_3)}{\det(\nabla u_1, \nabla u_2)} = \nabla \frac{H_{11} H_{23} - H_{12} H_{13}}{H_{11} H_{22} - H_{12}^2}.$$  

$X$ is also known from the data. One deduces

$$\tilde{A}^2 = \tilde{\gamma} = (JX \cdot Y)^{-1} J(XX^T + YY^T) J$$

If one can guarantee that $Y$ never vanishes over some $\Omega$, then $\tilde{\gamma}$ is uniquely reconstructed over $\Omega$ with a stability estimate of the form

$$\|\tilde{\gamma} - \tilde{\gamma}'\|_{L^\infty(\Omega)} \leq C \|H - H'\|_{W^{1,\infty}(\Omega)}.$$
Reconstruction of $|\gamma|^{\frac{1}{2}}$

Once $\tilde{A}$ or $\tilde{\gamma}$ is known (or reconstructed), use $u_1, u_2$ to reconstruct
- the angle function $\theta$ via the equation
  \[ \nabla \theta = \mathbf{V} - \tilde{A}^{-2} \left( \frac{1}{2} \mathbf{JN} + [\tilde{A}_2, \tilde{A}_1] \right) \]
- the function $|\gamma|^{\frac{1}{2}}$ via its gradient equation.

**Stability:** using integration along curves and Gronwall’s lemma, one obtains **Lipschitz stability** in $W^{1,\infty}$ for $\theta$ and $\log |\gamma|^{\frac{1}{2}}$.

\[ \| \log |\gamma|^{\frac{1}{2}} - \log |\gamma'|^{\frac{1}{2}} \|_{W^{1,\infty}(\Omega)} \leq C \| H - H' \|_{W^{1,\infty}(\Omega)}. \]
Requirements for the algorithm

Two crucial assumptions:

- \( \inf_{\Omega} \det(\nabla u_1, \nabla u_2) \geq c_1 > 0. \)
- \( \nabla \left( \frac{\det(\nabla u_1, \nabla u_3)}{\det(\nabla u_1, \nabla u_2)} \right) \neq 0, \quad x \in \Omega. \)

Q: How to control this with the boundary inputs \((g_1, g_2, g_3)\)?
- Do such \((g_1, g_2, g_3)\) exist?
- Do we have an explicit expression?
What boundary conditions work? 1/2

Question 1: how do we find $g_1, g_2$ such that
\[ \inf_X \det(\nabla u_1, \nabla u_2) \geq c_0 > 0 \? \]

- If $(g_1, g_2)$ is a homeomorphism of $\partial X$ onto its image, then the above condition holds [Alessandrini-Nesi, ’01]. Take $(g_1, g_2) = Id_X$!

- Isotropic case: traces of CGO solutions ensure this for $|\rho|$ large enough.

- Anisotropic case: use the isotropic representative, define CGO’s and push-forward.
What boundary conditions work? 2/2

**Question 2:** how do we find $g_1, g_2, g_3$ such that $Y$ vanishes as rarely as possible?

- **Isotropic case:** CGO solutions
- **Anisotropic case:** the vector $Y$ transforms nicely with push-forwards so that we can use the CGO construction of the corresponding isotropic metric.
- 3 linearly independent $(g_1, g_2, g_3)$ should suffice to achieve this almost everywhere.
Complex Geometric Optics solutions

For $\gamma$ isotropic, these are complex-valued solutions of the form
$$u_\rho = \frac{1}{\sqrt{\gamma}} e^{\rho \cdot x} (1 + \phi_\rho),$$
where $\rho \cdot \rho = 0$.

This implies $\rho = r(k + ik_\perp)$ for some $k \in \mathbb{S}^1$ and $k_\perp = Jk$.
Moreover, we have $r\phi_\rho = O(1)$.

Answer to question 1: take $(u_1, u_2) = (\Re(u_{\rho_1}), \Im(u_{\rho_1}))$ for some $\rho_1 = r(k_1 + ik_\perp)$, then we have that
$$\det(\nabla u_1, \nabla u_2) \approx r + O(1).$$

Answer to question 2: take $(u_1, u_2, u_3, u_4) = (\Re(u_{\rho_1}), \Im(u_{\rho_1}), \Re(u_{\rho_2}), \Im(u_{\rho_2}))$ for some $\rho_1 = r(k_1 + ik_\perp)$, $\rho_2 = r(k_2 + ik_\perp)$ with $k_1 \neq k_2$, then we have
$$Y \approx r(k_1 - k_2) + O(1).$$

Requires regularity on $\gamma$. 
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Numerics

Coded in **MatLab** on a cartesian, equispaced grid, using second-order centered **finite differences**.

Decompose $\gamma = |\gamma|^{\frac{1}{2}} \tilde{\gamma}$ with
\[
\tilde{\gamma}(\xi,\zeta) = \begin{bmatrix}
\xi & \zeta \\
\zeta & \frac{1+\zeta^2}{\xi}
\end{bmatrix}
\] (det $\tilde{\gamma} = 1$).

Compute:

- **Solutions** $(u_1, u_2, u_3)$ with $(g_1, g_2, g_3)(x, y) = (x + y, y + 0.1y^2, -x + y)$
- **Data** $H_{ij} = \nabla u_i \cdot \gamma \nabla u_j$ with **noise** $H_{noisy} = H \ast (1 + \frac{\alpha}{100} \text{ random})$. 
Numerics - Data $H_{ij}$ and reconstruction of $|\gamma|$

Examples of power densities ($H_{11}$ and $H_{12}$)

$\alpha = 0\%$

Reconstruction of $|\gamma|$ (smooth and rough) with known $\tilde{\gamma}$

$\alpha = 30\%$

From noisy data ($\alpha = 30\%$)

true
Numerics - reconstruction of $\tilde{\gamma}(\xi, \zeta)$, then det $\gamma$

Anisotropy reconstruction formula:

$$\tilde{\gamma} = (JX \cdot Y)^{-1} J(XX^T + YY^T)J, \quad Y = \nabla \log \frac{H_{11}H_{23} - H_{12}H_{13}}{H_{11}H_{22} - H_{12}^2}.$$

true $(\xi, \zeta)$

with rough $|\gamma|$ and $\alpha = 0\%$

with smooth $|\gamma|$ and $\alpha = 0.1\%$

det $\gamma$, true and recons. ($\alpha = 0.1\%$)
Numerics - reconstruction of $\tilde{\gamma}$

Lack of robustness to noise of the vector field $Y = \nabla(d_1/d_2)$. One may tackle this problem by adding measurements and minimizing a least-squares problem for $(\xi, \zeta)$.

(e) $d_1/d_2, \alpha = 0\%$ (f) $d_1/d_2, \alpha = 1\%$ (g) $\xi$ at $\{x = 0.5\}$ (h) $\zeta$ at $\{x = 0.5\}$

Figure: (e)&(f): influence of the noise on the function $d_1/d_2$. (g)&(h): cross sections of $\xi$ and $\zeta$ using least-square based reconstruction formulas.
Remarks

Power densities provide

- full inversion formulas,
- Lipschitz stability for $|\gamma|^{\frac{1}{2}}$, with great robustness to noise: resolution on $|\gamma|^{\frac{1}{2}}$ is as good as that on the $H_{ij}$’s.
- Hölder stability for $\tilde{\gamma}$ with much less robustness to noise.
- conditions of validity of these algorithms easy to fulfill.

Generalization to higher dimensions.

- reconstruction of $|\gamma|$ works and is Lipschitz-stable.
- reconstruction formulas for $\tilde{\gamma}$ have been derived with the same type of stability.
- unlike in two dimensions, the conditions of validity of these algorithms might not always be globally fulfilled.

[Bal-Courdurier, ’13]
Thank you!

References available at
http://www.math.washington.edu/~fmonard/research.html