Inverse anisotropic conductivity from power densities in two dimensions

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Outline

Overview

Inverse problem: Find $x \in \mathfrak{X}$ such that $y = \mathfrak{M}(x)$,
given **data** $y \in \mathfrak{Y}$ and **model** $\mathfrak{M}$.

Examples:

X-Ray CT: a function $\overset{?}{\Leftarrow}$ its line integrals. $\mathfrak{M}$: X-Ray transform.

Calderón's problem (EIT): $\gamma \overset{?}{\Leftarrow} \Lambda_\gamma$, $\mathfrak{M}$: conductivity eq.

Seismography: sound speed in Earth's crust $\overset{?}{\Leftarrow}$ wave traveltimes.

Inverse wave problem: $u|_{t=0} \overset{?}{\Leftarrow} u|_{\partial X \times [0,T]}$, $\mathfrak{M}$: wave eq.

Analysis of inverse problems

The theoretical analysis of an inverse problem consists in

- **Uniqueness** assessment: $\mathfrak{M}(x_1) = \mathfrak{M}(x_2) \overset{?}{\Longrightarrow} x_1 = x_2$
- **Stability** assessment (rules resolution in practice)
  Hilbert scale of stability:
  - well-posed problems ($\delta\mathfrak{M}$ small $\Longrightarrow$ δx small)
  - mildly ill-posed problems ($\delta\mathfrak{M}$ small $\Longrightarrow$ δx not too large)
  - severely ill-posed ($\delta\mathfrak{M}$ small $\Longrightarrow$ δx too large)
- Derivation of **reconstruction formulas**
- Numerical implementation/simulation

$\triangleright$ Applications to **imaging sciences**
(medical, geophysical, atmospheric, etc. . .).

Two (linear) examples

1. $\mathfrak{M}_1[f](x) = \int_0^x f(t)\ dt$.
2. $\mathfrak{M}_2[f](x) = u(x,T)$ ($T > 0$ fixed), where u solves

$$\partial_t u = \partial_{xx} u \quad (t > 0, x \in \mathbb{R}), \qquad u(x,0) = f(x).$$

Both problems are injective. In a noiseless world, $f(x) = \frac{d}{dx}\mathfrak{M}_1[f](x)$, and in particular,

$$\|f - g\|_{L^2(\mathbb{R})} \leq \|\mathfrak{M}_1[f] - \mathfrak{M}_1[f]\|_{H^1(\mathbb{R})}.$$

For the second problem, $f(x) = \mathcal{F}^{-1}_{\xi \to x}(e^{T\xi^2}\mathcal{F}_{x\to\xi}\mathfrak{M}_2[f])$, though there is no (s,p,C) such that

$$\|f - g\|_{H^s(\mathbb{R})} \leq C\|\mathfrak{M}_2[f] - \mathfrak{M}_2[g]\|_{H^{s+p}(\mathbb{R})}.$$

Problem 1 is mildly ill-posed while problem 2 is severely ill-posed.

▷ Stability quantifies the resolution available on reconstructions.

Motivation

Underlying goal: To **improve resolution** in soft tissue medical imaging modalities (e.g. from centimetric to millimetric).

- **Mechanical**, **optical** and **electrical** properties of tissues display **good contrast** for e.g. tumor detection, lung activity monitoring.
- Their corresponding imaging modalities are very **poorly resolved**.
- Inverse problems: mathematically **severely ill-posed**.

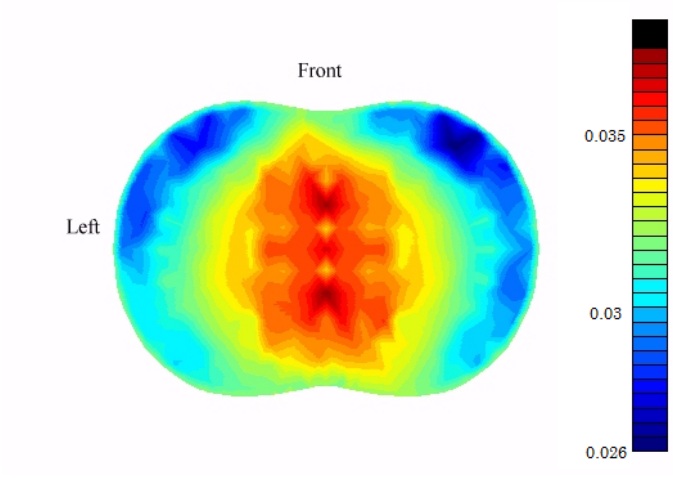


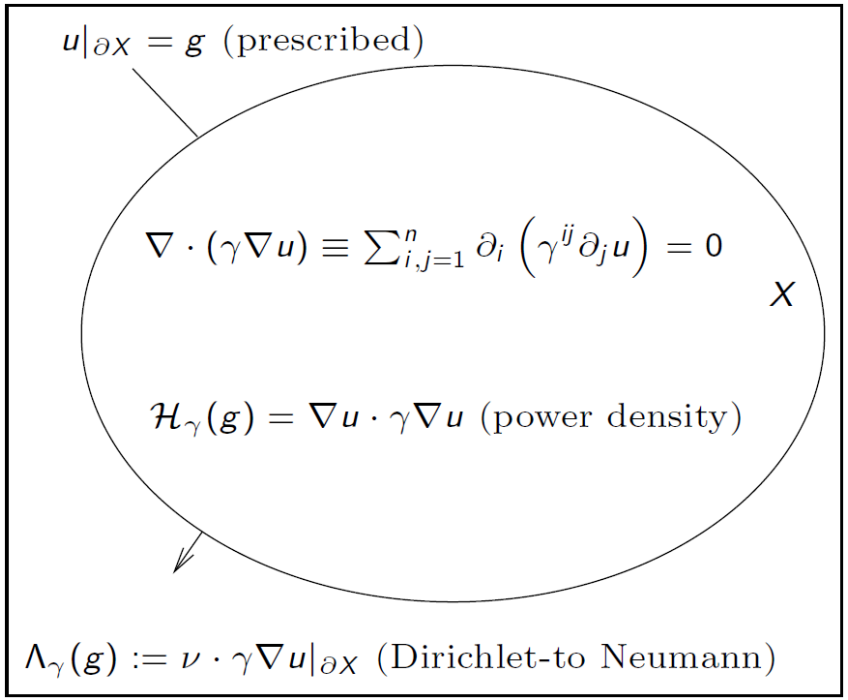
Figure : Conductivity map of human chest

[Kerrouche et al. '01]

Outline

The inverse conductivity problem

The model: $X \subset \mathbb{R}^n$ bounded domain.



γ : *uniformly elliptic* conductivity tensor.

- **Calderón's problem:**
Does Λ_γ determine γ uniquely ? stably ?

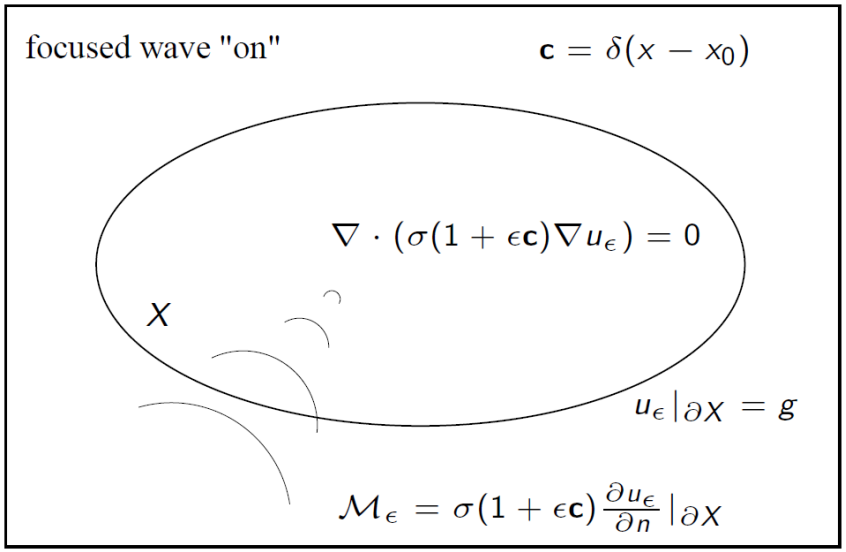
[Calderón '80]

- **Power density problem:**
Does $\mathcal{H}_\gamma$ determine γ uniquely ? stably ?

Note: $\int_X \mathcal{H}_\gamma u = \int_{\partial X} g \Lambda_\gamma g$

Derivation of power densities - 1/2

By ultrasound modulation



Physical focusing

[Ammari et al. '08]

Synthetic focusing

[Kuchment-Kunyansky '10]

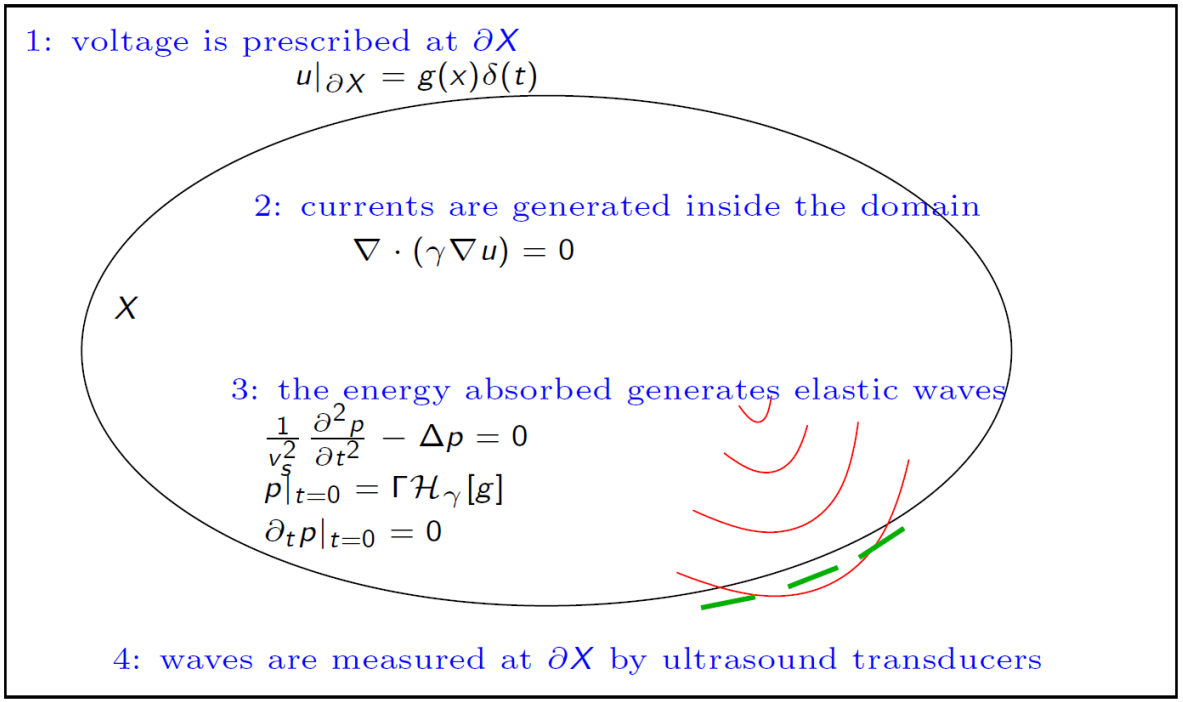
[Bal-Bonnetier-M.-Triki '11]

Small perturbation model:

$\frac{(\mathcal{M}_\epsilon - \mathcal{M}_0)}{\epsilon}$ gives an approximation of $\nabla u_0 \cdot \gamma \nabla u_0$ at x_0.

Derivation of power densities - 2/2

By thermoelastic effects (Impedance-Acoustic CT)



One reconstructs $\Gamma \mathcal{H}_\gamma = \Gamma \nabla u \cdot \gamma \nabla u$ over X (Γ: Grüneisen coefficient)

[Gebauer-Scherzer '09]

Theoretical comparison of inversions

From Dirichlet-to-Neumann map (latest 2D results):

- **uniqueness:** holds for isotropic L^∞ [Astala-Päivärinta '05]. Anisotropic case: lack of injectivity completely characterized [Astala-Lassas-Päivärinta '05].
- **stability:** logarithmic, i.e. *severely ill-posed*. Poor resolution in practice. [Alessandrini '88], [Santacesaria '11]

From ("enough") power density measurements:

Define $|\gamma| := \det \gamma$ and $\tilde{\gamma} := |\gamma|^{-\frac{1}{2}}\gamma$ ($\tilde{\gamma}$ satisfies $\det \tilde{\gamma} = 1$).

- **uniqueness:** both $|\gamma|^{\frac{1}{2}}$ and $\tilde{\gamma}$ are uniquely reconstructible via explicit, algebraic algorithms
- **stability:** well-posed (Lipschitz) in $W^{1,\infty}$ for $|\gamma|^{\frac{1}{2}}$, ill-posed of order 1 (Hölder) for $\tilde{\gamma}$.

Power density measurements - References

References on resolution of the power density problem:

- 2D isotropic [Capdeboscq et al. '09].
- 2D-3D isotropic linearized [Kuchment-Kunyansky '11].
- 2D-3D isotropic [Bal-Bonnetier-M.-Triki,IPI '12].
- nD isotropic: [M.-Bal, IPI '12] [Kocyigit '12]
- 2D anisotropic: [M.-Bal, IP '12] (today's talk)
- nD anisotropic: [M.-Bal, CPDE '13].

Pseudodifferential analysis on the linearized problem:

- Isotropic case [Kuchment-Steinhauer, '12].
- nD anisotropic case [Bal-Guo-M., IPI '13].

Resolution from a single power density - isotropic case:

- Newton-based numerical methods to recover (u, γ) [Ammari et al. '08, Gebauer-Scherzer '09].
- Theoretical work on the Cauchy problem [Bal '11].

Outline

The problem

Problem: recover $\gamma = (|\gamma|^{\frac{1}{2}}, \tilde{\gamma})$ from the knowledge of $H_{ij}(x) = \gamma \nabla u_i \cdot \nabla u_j$, where u_i solves

$$\nabla \cdot (\gamma \nabla u_i) = 0, \quad u_i|_{\partial\Omega} = g_i, \quad 1 \leq i \leq 3.$$

"Reparameterize" the problem: Define $A = \gamma^{\frac{1}{2}}$ and $S_i := A \nabla u_i$, all unknowns. The S_i's satisfy

$$\nabla \cdot (A S_i) = 0, \quad (J\nabla) \cdot (A^{-1} S_i) = 0, \quad 1 \leq i \leq 3,$$

where $J := \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and the data takes the form $H_{ij} = S_i \cdot S_j$ (Grammian matrix).

Intermediate equations of interest

From two solutions (u_1, u_2): write a decomposition of $S = [S_1|S_2] = [A\nabla u_1|A\nabla u_2]$ of the form

$S = Q(\theta)R(H_{11}, H_{12}, H_{22})$ (e.g. "QR" or $S = Q(\theta)\begin{bmatrix} H_{11} & H_{12} \\ sym & H_{22} \end{bmatrix}^{\frac{1}{2}}$),

where $Q(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Using the previous PDE's, one derives:

$$\nabla \log |\gamma|^{\frac{1}{2}} = N + (\nabla H^{ij} \cdot \widetilde{A} S_i)\widetilde{A}^{-1} S_j,$$

$$\widetilde{A}^2 \nabla\theta + [\widetilde{A}_2, \widetilde{A}_1] = \widetilde{A}^2 V - \frac{1}{2} J N, \qquad \widetilde{A} := \tilde{\gamma}^{\frac{1}{2}},$$

where $V := \frac{1}{2}\frac{H_{11}}{d}\nabla\frac{H_{12}}{H_{11}}$ and $N := \nabla \log d$, with $d := (H_{11}H_{22} - H_{12}^2)^{\frac{1}{2}}$.

Legend: known data, anisotropic structure, unknown.

Reconstruction of $\tilde{\gamma}$

Using the above equation for **two** systems (u_1, u_2) and (u_1, u_3):

$$\widetilde{A}^2 X = JY, \quad Y := \nabla \frac{\det(\nabla u_1, \nabla u_3)}{\det(\nabla u_1, \nabla u_2)} = \nabla \frac{H_{11}H_{23} - H_{12}H_{13}}{H_{11}H_{22} - H_{12}^2}.$$

X is also known from the data. One deduces

$$\widetilde{A}^2 = \tilde{\gamma} = (JX \cdot Y)^{-1} J(XX^T + YY^T)J$$

If one can guarantee that Y never vanishes over some Ω, then $\tilde{\gamma}$ is uniquely reconstructed over Ω with a stability estimate of the form

$$\|\tilde{\gamma} - \tilde{\gamma}'\|_{L^\infty(\Omega)} \leq C\|H - H'\|_{W^{1,\infty}(\Omega)}.$$

Reconstruction of $|\gamma|^{\frac{1}{2}}$

Once $\widetilde{A}$ or $\tilde{\gamma}$ is known (or reconstructed), use u_1, u_2 to reconstruct

- the angle function θ via the equation

$$\nabla\theta = V - \widetilde{A}^{-2}\left(\frac{1}{2}JN + [\widetilde{A}_2, \widetilde{A}_1]\right)$$

- the function $|\gamma|^{\frac{1}{2}}$ via its gradient equation.

Stability: using integration along curves and Gronwall's lemma, one obtains **Lipschitz stability** in $W^{1,\infty}$ for θ and $\log|\gamma|^{\frac{1}{2}}$.

$$\| \log|\gamma|^{\frac{1}{2}} - \log|\gamma'|^{\frac{1}{2}} \|_{W^{1,\infty}(\Omega)} \leq C\|H - H'\|_{W^{1,\infty}(\Omega)}.$$

Requirements for the algorithm

Two crucial assumptions:

- $\inf_{\Omega} \det(\nabla u_1, \nabla u_2) \geq c_1 > 0.$
- $\nabla(\det(\nabla u_1, \nabla u_3)/\det(\nabla u_1, \nabla u_2)) \neq 0, \quad x \in \Omega.$

Q: How to control this with the boundary inputs (g_1, g_2, g_3) ?

- Do such (g_1, g_2, g_3) exist ?
- Do we have an explicit expression ?

What boundary conditions work ? 1/2

Question 1: how do we find g_1, g_2 such that $\inf_X \det(\nabla u_1, \nabla u_2) \geq c_0 > 0$?

- If (g_1, g_2) is a homeomorphism of ∂X onto its image, then the above condition holds [Alessandrini-Nesi, '01]. Take $(g_1, g_2) = Id_X$!
- isotropic case: traces of CGO solutions ensure this for $|\rho|$ large enough.
- anisotropic case: use the isotropic representative, define CGO's and push-forward.

What boundary conditions work ? 2/2

Question 2: how do we find g_1, g_2, g_3 such that Y vanishes as rarely as possible ?

- isotropic case: CGO solutions
- anisotropic case: the vector Y transforms nicely with push-forwards so that we can use the CGO construction of the corresponding isotropic metric.
- 3 linearly independent (g_1, g_2, g_3) should suffice to achieve this almost everywhere.

Complex Geometric Optics solutions

For γ isotropic, these are complex-valued solutions of the form $u_\rho = \frac{1}{\sqrt{\gamma}} e^{\rho \cdot x}(1 + \phi_\rho)$, where $\rho \cdot \rho = 0$.
This implies $\rho = r(k + ik^\perp)$ for some $k \in \mathbb{S}^1$ and $k^\perp = Jk$.
Moreover, we have $r\phi_\rho = O(1)$.
Answer to question 1: take $(u_1, u_2) = (\Re(u_{\rho_1}), \Im(u_{\rho_1}))$ for some $\rho_1 = r(k_1 + ik_1^\perp)$, then we have that

$$\det(\nabla u_1, \nabla u_2) \approx r + O(1).$$

Answer to question 2: take $(u_1, u_2, u_3, u_4) = (\Re(u_{\rho_1}), \Im(u_{\rho_1}), \Re(u_{\rho_2}), \Im(u_{\rho_2}))$ for some $\rho_1 = r(k_1 + ik_1^\perp)$, $\rho_2 = r(k_2 + ik_2^\perp)$ with $k_1 \neq k_2$, then we have

$$Y \approx r(k_1 - k_2) + O(1).$$

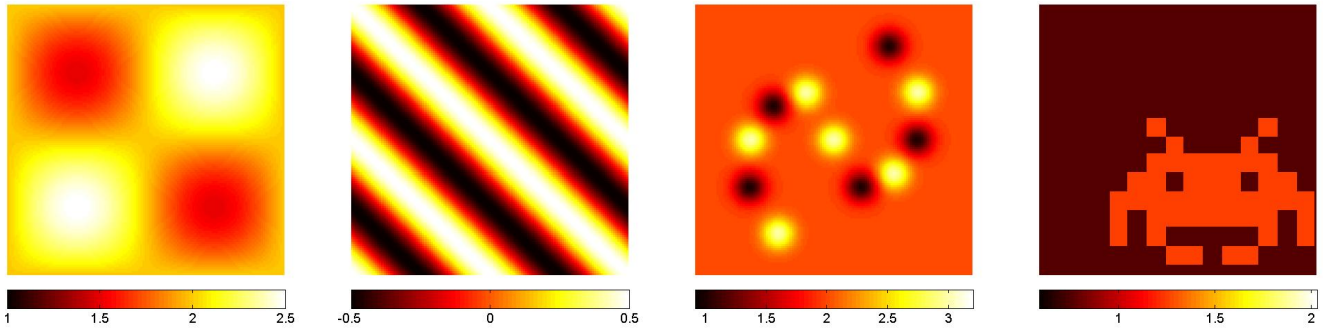
Requires regularity on γ.

Outline

Numerics

Coded in **MatLab** on a cartesian, equispaced grid, using second-order centered **finite differences**.

Decompose $\gamma = |\gamma|^{\frac{1}{2}}\ \tilde{\gamma}$ with $\tilde{\gamma}(\xi,\zeta) = \begin{bmatrix} \xi & \zeta \\ \zeta & \frac{1+\zeta^2}{\xi} \end{bmatrix}$ $(\det\tilde{\gamma} = 1)$.



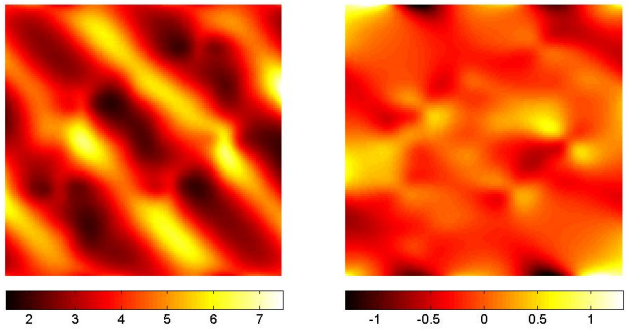
(a) ξ (b) ζ (c) $|\gamma|^{\frac{1}{2}}$ smooth (d) $|\gamma|^{\frac{1}{2}}$ rough

Compute:

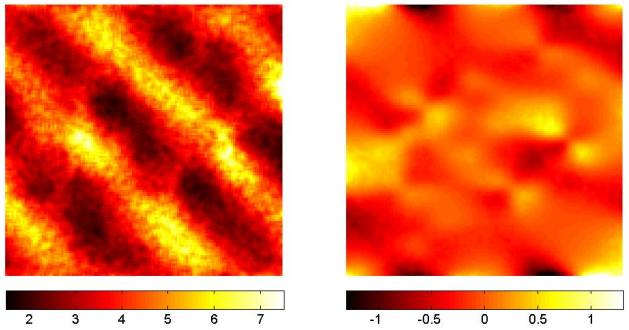
- Solutions (u_1, u_2, u_3) with $(g_1, g_2, g_3)(x, y) = (x+y, y+0.1y^2, -x+y)$
- Data $H_{ij} = \nabla u_i \cdot \gamma \nabla u_j$ with **noise** $H_{noisy} = H.*\ (1 + \frac{\alpha}{100}\ \texttt{random})$.

Numerics - Data H_{ij} and reconstruction of $|\gamma|$

Examples of power densities (H_{11} and H_{12})

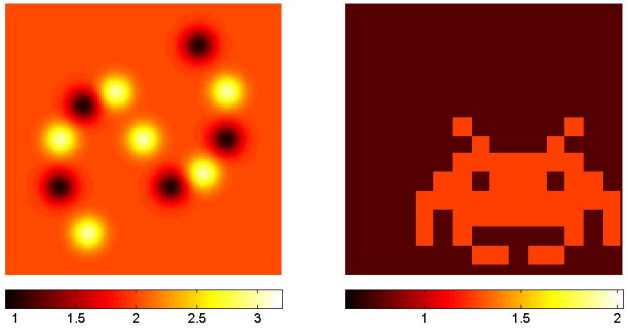


$\alpha = 0\%$

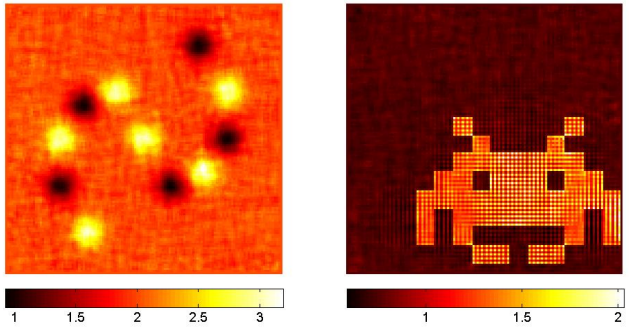


$\alpha = 30\%$

Reconstruction of $|\gamma|$ (smooth and rough) with known $\tilde{\gamma}$



true

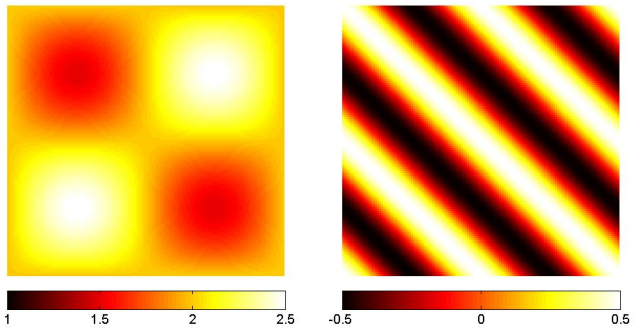


From noisy data ($\alpha = 30\%$)

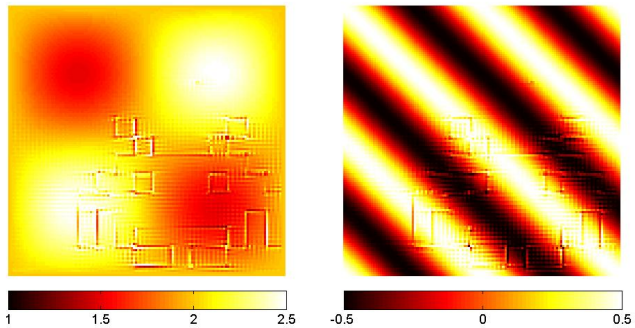
Numerics - reconstruction of $\tilde{\gamma}(\xi,\zeta)$, then $\det\gamma$

Anisotropy reconstruction formula:

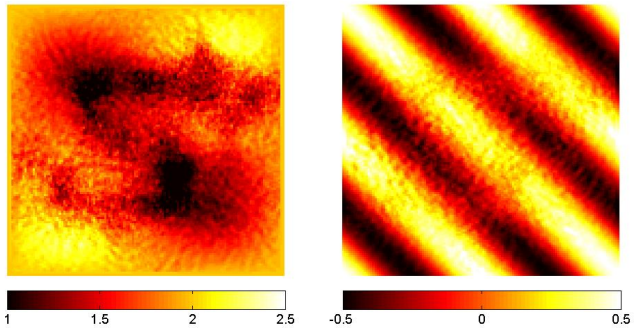
$$\tilde{\gamma} = (JX\cdot Y)^{-1}J(XX^T + YY^T)J, \quad Y = \nabla \log \frac{H_{11}H_{23} - H_{12}H_{13}}{H_{11}H_{22} - H_{12}^2}.$$



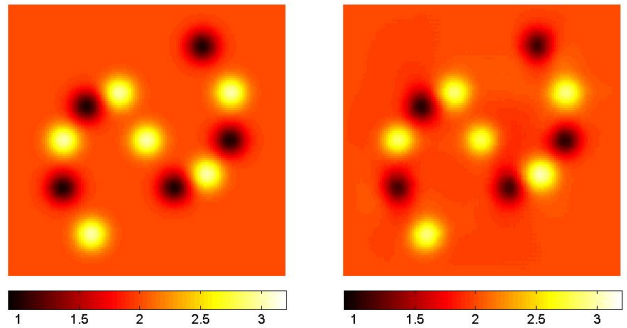
true (ξ,ζ)



with rough $|\gamma|$ and $\alpha = 0\%$



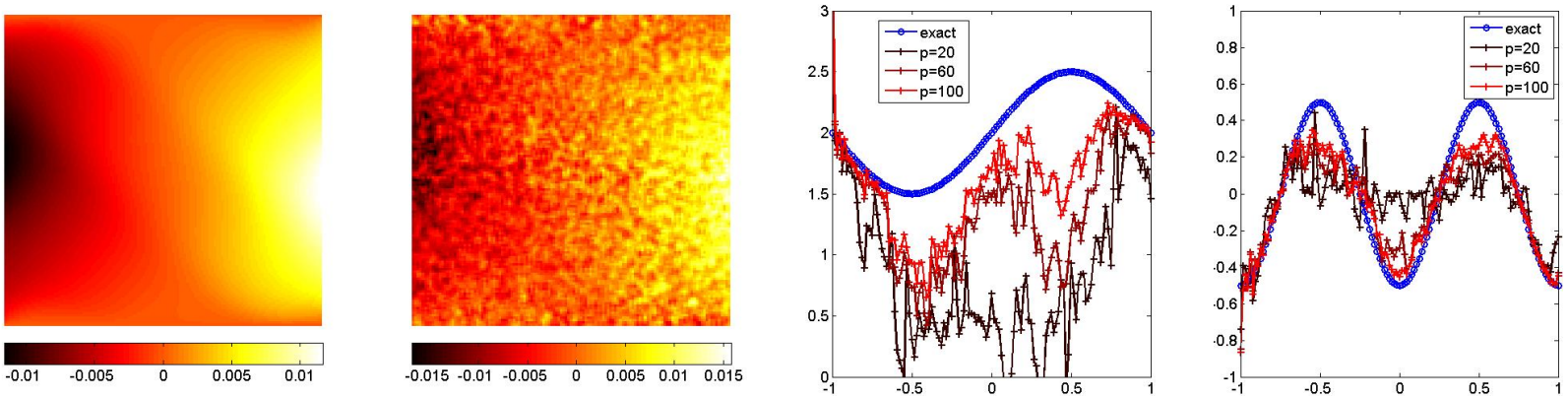
with smooth $|\gamma|$ and $\alpha = 0.1\%$



$\det\gamma$, true and recons. $(\alpha = 0.1\%)$

Numerics - reconstruction of $\tilde{\gamma}$

Lack of robustness to noise of the vector field $Y = \nabla(d_1/d_2)$. One may tackle this problem by adding measurements and minimizing a least-squares problem for (ξ, ζ).



(e) $d_1/d_2, \alpha = 0\%$ (f) $d_1/d_2, \alpha = 1\%$ (g) ξ at $\{x = 0.5\}$ (h) ζ at $\{x = 0.5\}$

Figure : (e)&(f): influence of the noise on the function d_1/d_2. (g)&(h): cross sections of ξ and ζ using least-square based reconstruction formulas.

Remarks

Power densities provide

- full inversion formulas,
- Lipschitz stability for $|\gamma|^{\frac{1}{2}}$, with great robustness to noise: resolution on $|\gamma|^{\frac{1}{2}}$ is as good as that on the H_{ij}'s.
- Hölder stability for $\tilde{\gamma}$ with much less robustness to noise.
- conditions of validity of these algorithms easy to fulfill.

Generalization to higher dimensions.

- reconstruction of $|\gamma|$ works and is Lipschitz-stable.
- reconstruction formulas for $\tilde{\gamma}$ have been derived with the same type of stability.
- unlike in two dimensions, the conditions of validity of these algorithms might not always be globally fulfilled.
  [Bal-Courdurier, '13]

Thank you !

References available at
http://www.math.washington.edu/~fmonard/research.html