Recent progress on the explicit inversion of geodesic X-ray transforms

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Outline

1 Introduction

2 Preliminaries and past results

3 Fredholm equations for $k$-differentials

4 Numerical simulations
Geodesic X-Ray transforms in two dimensions

\((M, g)\) non-trapping Riemannian manifold with boundary with unit tangent bundle \(TM\) and influx boundary

\[ \partial_+ TM = \{(x, v) \in TM : x \in \partial M, g(v, \nu_x) > 0\}, \quad \nu_x : \text{unit inner normal} \]

Geodesic flow: \(\phi_t(x, v) = (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))\) for \((x, v) \in TM, \, t \in [\tau(x, -v), \tau(x, v)]\).

Geodesic X-ray transform of a sym. cov. \(m\)-tensor \(f\):

\[ If(x, v) = \int_0^{\tau(x, v)} f(\gamma_{x,v}(t), \dot{\gamma}^m_{x,v}(t)) \, dt, \quad (x, v) \in \partial_+ TM. \]

Restrict to unit-speed geodesics by homogeneity arguments. Obstruction to injectivity: if \(f\) is a symmetric \(m\)-tensor of the form \(f = \sigma \nabla h\) with \(h\) an \(m - 1\)-tensor vanishing at the boundary, then \(If = 0\).
Explicit inversion of geodesic X-ray transforms

Introduction

Inverse Problems with $gXrt$

S-injectivity question: is the ray transform injective over solenoidal tensors, i.e. modulo this natural obstruction?

Sharafutdinov’s decomp.: $f = \sigma \nabla h + f^s$ with $\delta f^s = 0$

$\delta := (-\sigma \nabla)^*.$

Applications/connection with other inverse geometric problems:

- X-ray CT, inverse transport: ($m = 0$)
- Ultrasound Doppler tomography: ($m = 1$)
- Deformation (i.e. linearized) boundary rigidity: conformal perturbation ($m = 0$) or full perturbation ($m = 2$).
- travel time tomography in slightly anisotropic elastic media ($m = 4$).  [Sharafutdinov ’94]
Literature 1/2 (Restricted to 2D) - simple case

Classically: [Herglotz 1905], [Radon ’17], [Funk ’16], [Helgason ]

Injectivity:
- Injectivity over functions [Mukhometov ’75]
- S-injectivity over vector fields [Anikonov–Romanov ’97]
- S-injectivity over 2-tensors [Sharafutdinov ’07]
- S-injectivity over tensor fields [Dairbekov ’06] (assumptions on curvature), [Paternain–Salo–Uhlmann ’13] (simple metrics)

Stability:
- over tensor fields in the simple case: [Stefanov–Uhlmann ’04]

Reconstruction algorithms:
- [Sharafutdinov ’94]: $m \geq 0$, $n \geq 2$, Euclidean case.
- [Pestov–Uhlmann ’04]: Fredholm equations ($m = 0, 1$) on simple surfaces (+ range characterization)
- [Krishnan ’10]: $\|\nabla\kappa\|_\infty$ small $\implies$ equations are invertible.
Literature 2/2 - The non-simple case

S-Injectivity:
- [Sharafudtinov ’97] on non-trapping spherically symmetric layers \((n \geq 2)\).
- [Stefanov-Uhlmann ’08] real-analytic metrics satisfying additional conditions \((n \geq 3)\).
- [Uhlmann-Vasy ’13] local inj. on manifolds satisfying a certain foliation condition. \((n \geq 3)\).

Stability:
- [Stefanov-Uhlmann ’12] effect of fold caustics \((n \geq 2)\).
- [M.-Stefanov-Uhlmann, preprint], [Holman] general caustics.
1. Introduction

2. Preliminaries and past results

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The geometry of the unit sphere bundle 1/2

Define the unit circle bundle

$$SM := \{(x, v) \in TM : g(v, v) = 1\},$$

and denote by $X$ the geodesic vector field on $SM$. Let $V = \partial_\theta$ be the vertical derivative. One can generate a frame on $SM$ by constructing a third vector field $X_\perp := [X, V]$, giving rise to the additional structure equations

$$[X_\perp, V] = -X \quad \text{and} \quad [X, X_\perp] = \kappa V.$$

Expression in isothermal coordinates (write $g = e^{2\lambda}Id$):

$$X = e^{-\lambda}(\cos \theta \partial_x + \sin \theta \partial_y + (-\sin \theta \lambda_x + \cos \theta \lambda_y)\partial_\theta).$$

$$X_\perp = -e^{-\lambda}(-\sin \theta \partial_x + \cos \theta \partial_y - (\cos \theta \lambda_x + \sin \theta \lambda_y)\partial_\theta).$$

Define on $SM$ the metric $d\Sigma^3$ such that $(X, X_\perp, V)$ is orthonormal.
The geometry of the unit sphere bundle 2/2

The space $L^2(SM, d\Sigma^3)$ decomposes orthogonally into

$$L^2(SM) = \bigoplus_{k \in \mathbb{Z}} H_k, \quad H_k := \ker(V - ikI), \quad \Omega_k = H_k \cap C^\infty(SM).$$

An element $u(x, \theta) \in L^2(SM)$ may thus be written as

$$u = \sum_{k \in \mathbb{Z}} u_k, \quad u_k(x, \theta) := e^{ik\theta} \tilde{u}_k(x) = \frac{e^{ik\theta}}{2\pi} \int_{S^1} u(x, \theta') e^{-ik\theta'} \, d\theta'.$$

In this decomposition, define the fiberwise Hilbert transform $H$ by

$$Hu = \sum_{k \in \mathbb{Z}} -i \text{sign}(k) u_k, \quad \text{with} \quad \text{sign}(0) = 0.$$

Even/odd decomposition: $u = u_+ + u_-$ with $u_\pm := \sum_{k \text{even/odd}} u_k$.

Note that $X(\Omega_k) \subset \Omega_{k+1} + \Omega_{k-1}$, similarly for $X_\perp$. 

The Pestov-Uhlmann reconstruction formulas 1/2

Commutator formula \cite{Pestov-Uhlmann '05}

\[
[H, X]u = X\perp u_0 + (X\perp u)_0,
\]

Geodesic transport: Denote by \( u^f \) the solution of

\[
Xu = -f \quad (SM), \quad u|_{\partial_- SM} = 0,
\]

so that \( u|_{\partial_+ SM} = If \). Define the linear operator

\( Wf := (X\perp u^f)_0 \).

Fredholm formulas: \cite{Pestov-Uhlmann '04, Theorem 5.4}

\[
(Id + W^2)f = -(X\perp w^{(f)}_\psi)_0, \quad w^{(f)} := H(If)_-.
\]

Similarly, for a solenoidal vector field of the form \( X\perp h \) with \( h|_{\partial M} = 0 \), the same approach yields

\[
(Id + (W^*)^2)h = -(w^{(h)}_{\psi})_0, \quad w^{(h)} := H(I[X\perp h])_+,
\]

where \( W^* h := (u^{X\perp h})_0 \) is the \( L^2(M) \)-adjoint operator to \( W \).
The Pestov-Uhlmann reconstruction formulas 2/2

Proof of Fredholm equation for $f$:

Hit the transport eq. $Xu^f = -f$ with $H$ to get

$$HXu^f = 0 = X(Hu^f) + X_\perp u^f_0 + (X_\perp u^f)_0.$$ 

Derive a transport problem for $Hu^f_-$

$$X(Hu^f_-) = -Wf, \quad u|_{\partial_+ SM} = \frac{1}{2}H(If)^- := w^{(f)},$$

thus guaranteeing that $Hu^f_- = u^Wf + w^{(f)}_\psi$. Compose with $(X_\perp \cdot)_0$ to make appear

$$(X_\perp (Hu^f_-))_0 = W^2 f + (X_\perp w^{(f)}_\psi)_0.$$ 

Hitting the last transport equation with $H$ again and using the commutator, we obtain $(X_\perp (Hu^f_-))_0 = -f$ hence the result.

Proof for solenoidal vector fields is similar.
Results and consequences

Next: when the metric is simple, it is proved in [Pestov-Uhlmann ’04, Prop. 5.1] that the operators $W$ and $W^*$ are smoothing, therefore compact in $\mathcal{L}(L^2(M))$.

Consequences:

- both reconstruction formulas stably reconstruct the singularities of $f$ and $h$. Invertibility of both equations holds modulo the finite-dimensional spaces $\ker(Id + W^2)$ and $\ker(Id + (W^*)^2)$ of smooth ghosts.
- If the metric has constant curvature, then $W \equiv W^* \equiv 0$, so the formulas above are exact.

Improvement [Krishnan ’10]: An estimate of the form $\|W\|_{\mathcal{L}(L^2)} \leq C \|\nabla \kappa\|_{\infty}$ implies that if the metric is close enough to constant curvature, $W$ and $W^*$ are contractions. In particular, the kernels above are trivial and $f$ and $h$ can be exactly reconstructed via Neumann series.
Outline

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4. Numerical simulations
A generalization to $k$-differentials

**Problem:** Reconstruct an element of the form $f = \tilde{f}(x)e^{ik\theta}$ for some fixed $k \in \mathbb{Z}$ and $f$ a smooth function from knowledge of $If$. Note the transport equation

$$Xu = -\tilde{f}(x)e^{ik\theta}, \quad u|_{\partial_SM} = 0.$$

**Applications:**

- An $m$-tensor has the form $f = \sum_{k=-m}^{m} \tilde{f}_k(x)e^{ik\theta},$
- building-block nature of $I$ restricted to $\Omega_k$ : was used in [Paternain-Salo-Uhlmann ’13] to give a range characterization of geodesic ray transforms over $m$-tensors.
- Setting $v := u^f e^{-ik\theta}$, $v$ satisfies the transport equation with a unitary connection, see [Paternain-Salo-Uhlmann ’13]

$$Xv + ik(X_\perp \lambda)v = -\tilde{f}(x), \quad v|_{\partial SM} = 0.$$
Explicit inversion of geodesic X-ray transforms
Fredholm equations for $k$-differentials

Connection to solenoidal tensors: $\delta(\tilde{f}dz^m) = (\partial_z \tilde{f})dz^{m-1}$, i.e. $\tilde{f}dz^m$ if and only if $\tilde{f}$ is holomorphic.

Injectivity (M., ’14): The problem is (completely) injective over $\Omega_k$ and $X_{\perp}\Omega_k$ as a consequence of [PSU ’12].

Idea for inversion: Hilbert transform shifted in frequency:
For $u \in L^2(\mathbb{SM})$, define $H_{(k)} u := e^{ik\theta} H(e^{-ik\theta} u)$. In fact, we have

$$H_{(k)} u = \sum_{l \in \mathbb{Z}} -i \text{sign}(l - k) u_l.$$ 

The commutator formula derived in [Pestov–Uhlmann ’05] can be generalized to this type of Hilbert transform as follows

$$[H_{(k)}, X]u = X_{\perp}u_k + (X_{\perp}u)_k \quad \text{and} \quad [H_{(k)}, X_{\perp}]u = - Xu_k - (Xu)_k.$$ 

Proof: direct calculation in the decomposition of $L^2(\mathbb{SM})$. 
Fredholm equations

Along the same lines as the case $k = 0$, working with transport equations, hitting them with $H(k)$ and using the new commutator formula, one is able to derive the inversion formula

\[(Id + W_k^2)f = -(X_\perp w^{(f)}_\psi)_k, \quad w^{(f)} := H(k)(lf)\sigma_k,\]

where $\sigma_k = +/-$ is $k$ is odd/even, respectively, and where we have defined $W_k f = (X_\perp u^f)_k$, mapping $\Omega_k$ (smooth elements of $H_k$) into itself.

By duality over $H_k$, one can define $W_k^* h = (u^{X_\perp h})_k$, and reconstruct $h \in \Omega_k$ from knowledge of $I(X_\perp h)$ via the equation

\[(Id + (W_k^*)^2)h = -(w^{(h)}_\psi)_k, \quad w^{(h)} := H(k)(I(X_\perp h))_{-\sigma_k} .\]
Making explicit the kernel of $W_k$, one is able to show that

- When the metric is simple, $W_k$ is smoothing, hence compact as an element of $\mathcal{L}(H_k)$.

- In the constant curvature case, $W_k \equiv 0$ only if $\kappa = 0$, i.e. for $\kappa = \text{cst} \neq 0$, the Fredholm equations above may not be directly invertible.

- In the simple case, there exists constants $C, C'$ such that

$$
\| W_k \|_{\mathcal{L}(H_k)} \leq C \| \nabla \kappa \|_{\infty} + k C' \| \kappa \|_{\infty}.
$$

As a consequence, when $\| \kappa \|_{C^1}$ is small enough, $W_k$ is again a contraction, and the equations above are explicitly invertible via Neumann series.
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Setting and parameterization of the data space

Work in *isothermal coordinates*. Degrees of freedom:

- the isotropic metric,
- the boundary (domain is star-shaped w.r.t. \((0, 0)\) by construction).

**Fan-beam coordinates:** The data space is an equispaced discretized version of \(S^1 \times [-\frac{\pi}{2}, \frac{\pi}{2}]\).

Examples of phantoms and domains:
Examples of metrics/geodesics

Constant curvature (left: positive; right: negative).

Non-constant curvature (focusing lens model).
Implementation of the ray transforms

Example of forward data and Hilbert transforms.
Metric: constant positive curvature with $R = 1.2$. 
Explicit inversion of geodesic X-ray transforms

Numerical simulations

One-shot inversion - constant positive curvature

$$g(x, y) = \frac{4R^4}{(x^2+y^2+R^2)^2}.$$ \( R = 1.2 \) (left) and \( R = 2 \) (right).
One-shot inversion - constant positive curvature

\[ g(x, y) = \frac{4R^4}{(x^2+y^2+R^2)^2}. \quad R = 1.2 \text{ (left) and } R = 2 \text{ (right)}. \]
Explicit inversion of geodesic X-ray transforms

Numerical simulations

One-shot inversion - constant negative curvature

\[ g(x, y) = \frac{4R^4}{(x^2 + y^2 - R^2)^2}. \quad R = 2 \text{ (left) and } R = 1.2 \text{ (right)}. \]
Explicit inversion of geodesic X-ray transforms

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One-shot inversion - constant negative curvature

\[ g(x, y) = \frac{4R^4}{(x^2 + y^2 - R^2)^2}. \quad R = 2 \text{ (left) and } R = 1.2 \text{ (right)}. \]
Constant curvature. Non-simple case

(a) Phantom $f$ and domain (b) Sample geodesics for $g_{R,+}$ with $R = 1$

(c) Forward data (d) Pointwise error $|f - f_{rc}|$

Figure: Non-simple domain with constant positive curvature.
Explicit inversion of geodesic X-ray transforms

General (non-constant curvature) case

**Goal:** from the equation $f + \mathcal{K}f = AIf$, compute a finite number of terms of the Neumann series

$$f = \sum_{p=0}^{\infty} (-\mathcal{K})^p AIf = \sum_{p=0}^{\infty} (Id - A)^p AIf.$$ 

**Experiments:** Metric: “focusing lens” with parameter $0 \leq k < e$. 

$$g(x, y) = \exp \left( k \exp \left( -\frac{r^2}{2\sigma^2} \right) \right), \quad \sigma = 0.25, \quad r^2 = (x - 0.2)^2 + y^2.$$ 

$k = 0.3$ 

$k = 0.6$ 

$k = 1.2$
Explicit inversion of geodesic X-ray transforms
Numerical simulations

Reconstruction of a function (t. to b.: $k = 0.3, 0.6, 1.2$)
Explicit inversion of geodesic X-ray transforms

Numerical simulations

Recons. of a solenoidal vector field ($k = 0.3, 0.6, 1.2$)
Explicit inversion of geodesic X-ray transforms

Numerical simulations

Higher-order tensors

Constant (positive) curvature

Top to bottom: $I(f(x)e^{ik\theta})$ for $k = 3, 6, 10$. Right: error on $f$ after 10 iterations. Curvature $\kappa = 0.25$. 
Explicit inversion of geodesic X-ray transforms

Numerical simulations

Higher-order tensors

Constant (positive) curvature

Top to bottom: \( I(f(x)e^{ik\theta}) \) for \( k = 3, 6, 10 \). Right: error on \( f \) after 10 iterations. Curvature \( \kappa = 0.694 \).
lens model (simple vs non-simple)

Top to bottom: $I(f(x)e^{3i\theta})$ for increasing lens intensity (simple, slightly non-simple, highly non-simple). Right: error on $f$ after 10 iterations.
Observation: artifacts appear at the conjugate locus of the initial “singularities”.

Joint work with Plamen Stefanov and Gunther Uhlmann.
Lack of stability in the presence of conjugate points

Suppose $p_1, p_2$ are conjugate along $\gamma_0$ and let $f_1$ supported near $p_1$ such that $(p_1, \xi_1) \in WF(f_1)$. Then there exists $f_2$ supported near $p_2$ with $(p_2, \xi_2) \in WF(f_2)$ such that $I(f_1 - f_2) \in C^\infty$.

In other words, the data $I(f_1 - f_2)$ does not allow us to resolve the singularities at $\xi_1$ or $\xi_2$, and the inversion problem becomes severely ill-posed.
Illustration of cancellation of singularities in the presence of caustics 1/2.
Explicit inversion of geodesic X-ray transforms

Numerical simulations

Higher-order tensors

Illustration of cancellation of singularities in the presence of caustics 2/2.

(d) $f_1 - f_2$ (right: with geodesics)

(e) $I f_1$

(f) $I(f_1 - f_2)$
Concluding remarks

The reconstruction algorithms from [Pestov-Uhlmann ’04] and [Krishnan ’10] were generalized to elements of the form $f$ and $X \perp h$, where $f, h$ are now $k$-differentials, gaining some understanding on inversion of solenoidal tensors and ray transforms with unitary connection.

**Numerics:**

- These reconstruction algorithms were successfully implemented in their context of validity, and the Neumann series converges even in some non-simple cases.
- Question: How to sample $\partial_+ SM$ appropriately?
- Question: How to find reconstruction algorithms including a regularization parameter? I.e. find a formula reflecting $\mathcal{W}_b \star f = R^\#(w_b \overset{\mathcal{S}}{\star} Rf)$ in the Euclidean case [Natterer ’01].
Thank you

References:
