Inverse diffusion with power density measurements.

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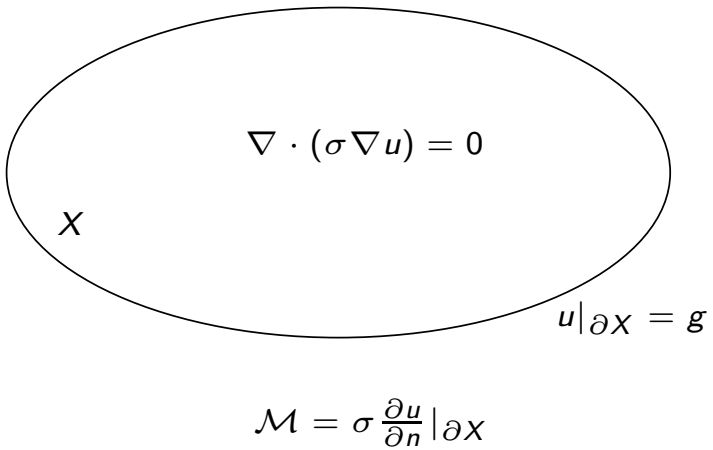
Inverse diffusion



Goal: Recover σ from prescribed g's and measured $\mathcal{M}$'s.

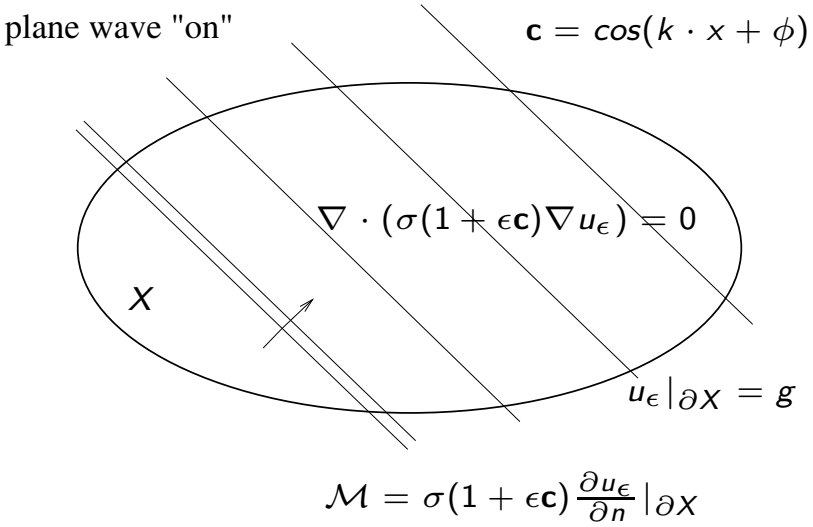
Applications:

- Electrical Impedance Tomography (conductivity equation), breast cancer detection,
- Optical Tomography (stationary diffusion).

Power densities

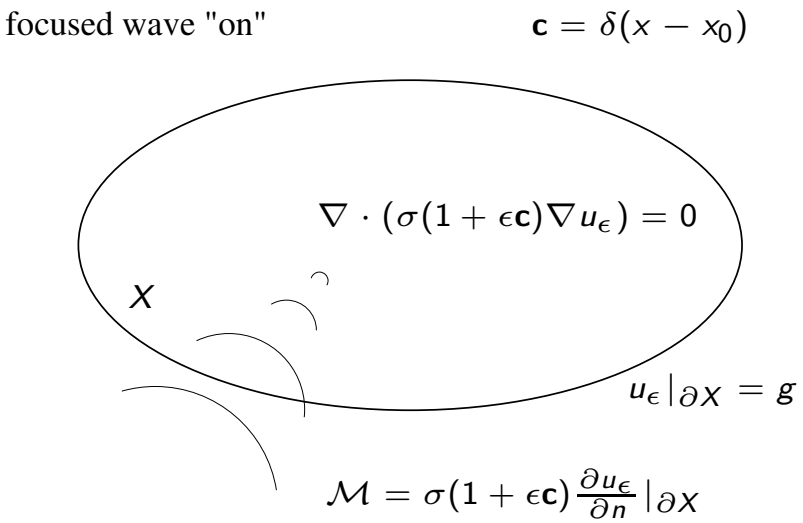
Measurements of the form $H(x) = \sigma(x)\nabla u(x) \cdot \nabla u(x)$, obtained by acoustic perturbations.

Synthesized focusing.



[KK '10, BMBT '11]

Physical focusing.



[Ammari et al. '08]

- OT and EIT have high contrast but poor resolution,
- pressure waves provide good resolution
- combination of the two should be promising.

Outline

Recalls on the old boundary value problem

On the inversion of the dirichlet-to-neumann map:

- Uniqueness for smooth σ [Calderón '80, Sylvester-Uhlmann '87]
- Inversion is severly ill-posed. Logarithmic stability: errors on reconstructions are bounded by $Ce^{\frac{1}{k}}$, where k is the detail size one wants to recovers.
- Consequence: very poor resolution. Practical applications reduced to mere anomaly detection. Still used for its high contrast.

Past work on the inversion

Work on power density measurements:

- Still open: study of the 0-Laplacian : (numerics on it: [Ammari et al '08])

$$\nabla \cdot \left(H \frac{\nabla u}{|\nabla u|^2} \right) = 0, \quad u|_{\partial X} = g.$$

- Capdebosq et al: 2D reconstruction theory. Numerics using minimization.
- Kuchment, Kunyanski '10: synthesized focusing, resolution using small perturbation model + numerics, 2D, 3D.

Outline

The problem

Problem: Recover the conductivity $\sigma \geq c_0 > 0$ from the knowledge of $H_{ij}(x) = \sigma \nabla u_i \cdot \nabla u_j$, where u_i solves

$$\nabla \cdot (\sigma \nabla u_i) = 0, \quad u_i|_{\partial X} = g_i, \quad 1 \leq i \leq 3.$$

A crucial assumption:

$$\inf_X (\det(\nabla u_1, \nabla u_2, \nabla u_3)) \geq c_1 > 0.$$

Set $S_i := \sqrt{\sigma} \nabla u_i$, and $F := \frac{1}{2} \nabla \log \sigma$. The S_i's satisfy the system

$$\nabla \cdot S_i = -F \cdot S_i, \quad \nabla \times S_i = F \times S_i,$$

and the data takes the form $H_{ij} = S_i \cdot S_j$.
Inversion for σ requires inverting for S and F.

The R frame

One can construct an everywhere oriented, orthonormal (i.e. $R^T R = I$, $\det R = 1$) frame $R_i = \sum_{j=1}^{3} t_{ij} S_j,\ 1 \leq i \leq 3$, e.g. by Gram-Schmidt. The R frame statisfies

$$\nabla \cdot R_i = -F \cdot R_i + V_{ik} \cdot R_k,$$
$$\nabla \times R_i = F \times R_i + V_{ik} \times R_k, \quad 1 \leq i \leq 3,$$

where the t_{ij}'s and V_{ij}'s depend on the data.
Overdetermined system: 12 scalar equations for 6 unknown scalar functions.

Resolution. Elimination of the source term

Since $R_i \times R_j = R_k$ for $(1,2,3) \to (i,j,k)$ a direct permutation, and we have the calculus identity

$$\nabla \cdot R_k = \nabla \cdot (R_i \times R_j) = R_j \cdot \nabla \times R_i - R_i \cdot \nabla \times R_j,$$

using these relations together with the R system allows us to derive an equation for F

$$F = \frac{1}{n}\left(\frac{1}{2}\nabla \log\det H + \sum_{1\le i,j\le n}((V_{ij}+V_{ji})\cdot R_i)R_j\right), \quad (n=3)$$

Allows us to close the system for the R_i's.

Resolution. First-order ODEs for R

Using more vector calculus allows us to derive a system of the form ($\mathbf{R} = (R_1^T, R_2^T, R_3^T)^T$)

$$\frac{\partial \mathbf{R}}{\partial x_k} = \sum_{|\alpha| \leq 3} Q_{\alpha,k} \mathbf{R}^\alpha, \quad 1 \leq k \leq 3,$$

where the functions $Q_{\alpha,k}$ only depend on the data. Still overdetermined, requires integrability conditions.

<u>Reconstruction scheme:</u>

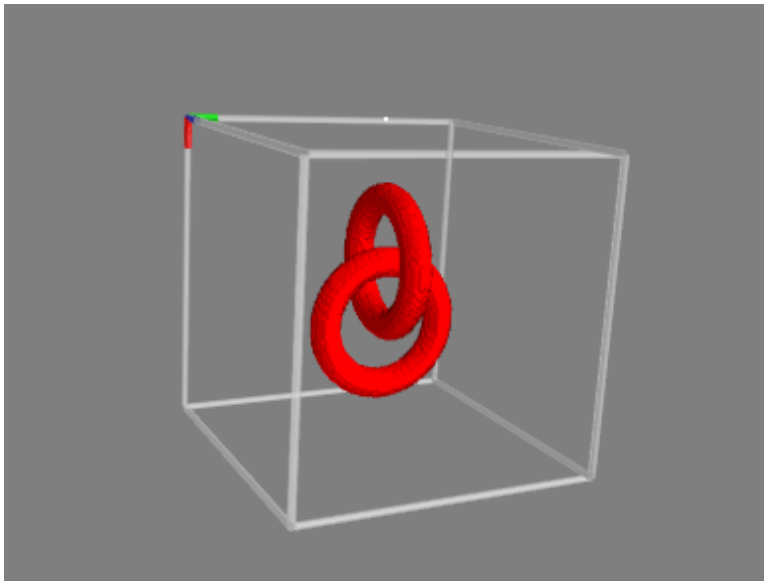
- ODE solving for the R_i's, then one knows F.
- reconstruct σ by integrating $\nabla \log \sigma = 2F$ along segments.

Choosing appropriate boundary inputs

How to choose boundary conditions so as to guarantee the condition $\det(\nabla u_1, \nabla u_2, \nabla u_3) = 0$?

- 2D: [Alessandrini, Nesi] the BCs $g_i = x_i, \ i = 1, 2$ work.
- 3D: [Briane, Milton and Nesi, 04'] there exist conductivities in 3D for which the determinant of the solutions changes sign even with the boundary conditions $g_i = x_i, 1 \leq i \leq 3$.

Counterexample in 3D:

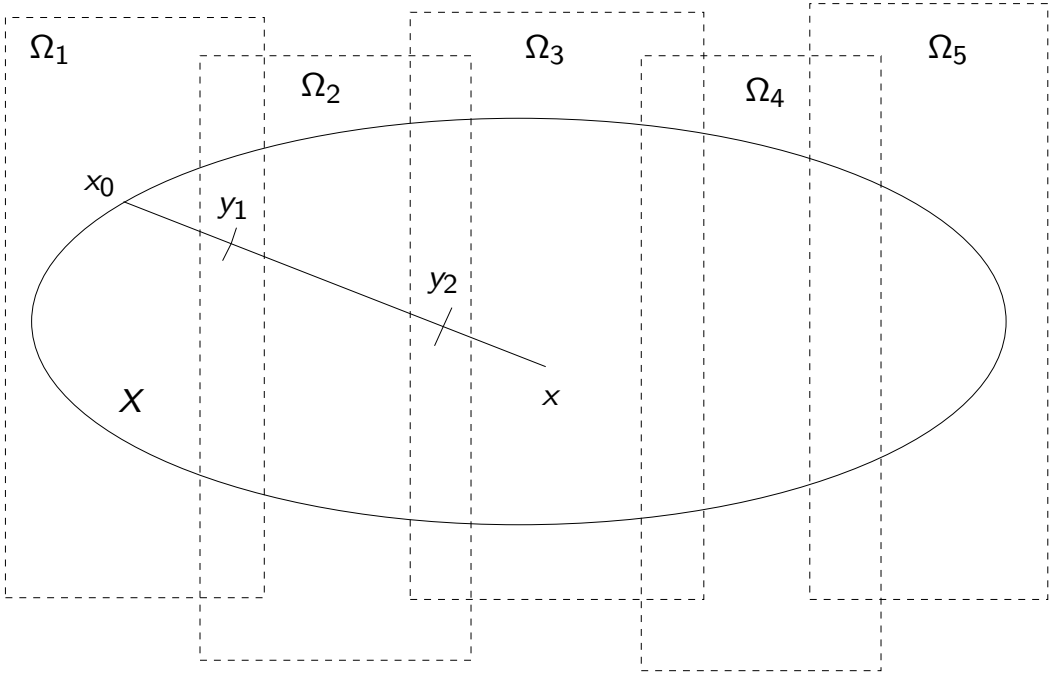


Global reconstruction

Solution: use 4 solutions that alternatively satisfy

$$\det(S_1, S_2, S_3) \geq c_0 > 0 \quad \text{on } \Omega_{2i},$$
$$\det(S_1, S_2, S_4) \geq c_0 > 0 \quad \text{on } \Omega_{2i+1},$$

for $1 \leq i \leq N$.



Global reconstruction

Construction of the solutions: uses the Complex Geometric Optics solutions first introduced by Calderón. Assuming $\sigma \in H^{\frac{7}{2}+\varepsilon}$, one can build solutions of $\nabla \cdot (\sigma \nabla u) = 0$ of the form

$$u_r = \sigma^{-\frac{1}{2}} e^{r(k+ik^\perp)\cdot x}(1 + \psi_r(x,k)),$$
$$k, k^\perp \in \mathbb{R}^3, k \cdot k^\perp = 0, \|\psi_r|_X\|_{\mathcal{C}^1} \leq \frac{C}{r}.$$

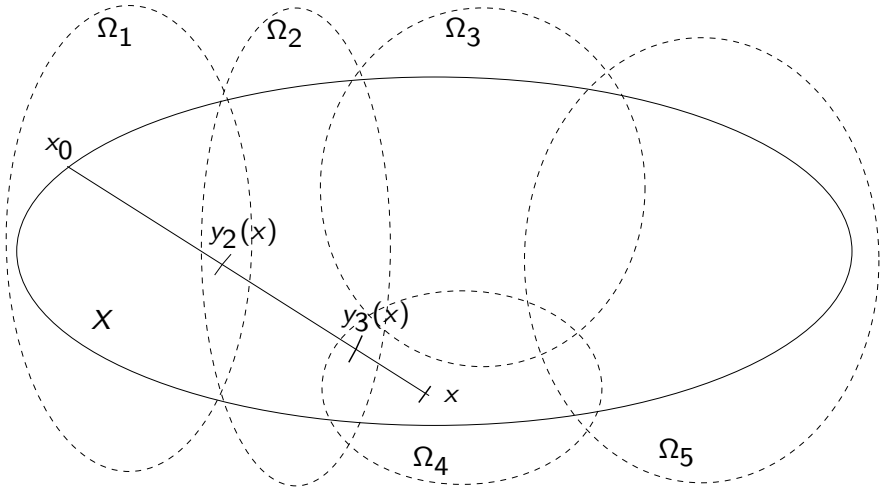
oscillates along $k^\perp$ and grows exponentially along k. For r large enough, we have

$$\Re(\sqrt{\sigma}\nabla u_r) \approx (k\cos(k^\perp \cdot x) - k^\perp \sin(k^\perp \cdot x))e^{k\cdot x},$$
$$\Im(\sqrt{\sigma}\nabla u_r) \approx (k^\perp \cos(k^\perp \cdot x) + k\sin(k^\perp \cdot x))e^{k\cdot x}.$$

One does not know ψ_ρ thus use approximate traces to define the g_i's.

Global reconstruction

More generally, one is able to deal with any situation of the form:



Stability

Theorem (Bal, M., Bonnetier, Triki 11')

$$\| \log \sigma - \log \tilde{\sigma} \|_{W^{1,\infty}(X)} \leq C\big(\epsilon_0 + \|H - \widetilde{H}\|_{W^{1,\infty}(X)}\big),$$

where ϵ_0 *is the error at the initial point* x_0

$$\epsilon_0 = |\log \sigma_0 - \log \tilde{\sigma}_0| + \sum_{i=1}^{4} \|S_i(x_0) - \widetilde{S}_i(x_0)\|.$$

This inversion is <u>well-posed</u>, i.e. resolution on σ is as good as that on the data H.

Conclusion

Ultrasound modulations in conjunction with high-contrast techniques offer promising improvements over regular OT and EIT:

- explicit reconstruction formula,
- well-posed inverse problems, therefore good resolution,
- local (i.e. requires partial knowledge at the boundary). Compatible with the fact that we cannot necessarily ensure 3 everywhere linearly independent solutions.

Future work:

- generalization to dimension n,
- numerical implementations in 3D,
- accounting for noise, characterizing the range of the measurement operator.