Inverse diffusion with power density measurements.

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Inverse diffusion

\nabla \cdot (\sigma \nabla u) = 0

\left. u \right|_{\partial X} = g

M = \sigma \frac{\partial u}{\partial n} \mid_{\partial X}

Goal: Recover \( \sigma \) from prescribed \( g \)'s and measured \( M \)'s.

Applications:
- Electrical Impedance Tomography (conductivity equation),
- breast cancer detection,
- Optical Tomography (stationary diffusion).
Power densities

Measurements of the form $H(x) = \sigma(x) \nabla u(x) \cdot \nabla u(x)$, obtained by acoustic perturbations.

Synthesized focusing.

Physical focusing.

plane wave "on" $c = \cos(k \cdot x + \phi)$

focused wave "on" $c = \delta(x - x_0)$

\[ \nabla \cdot (\sigma(1 + \epsilon c) \nabla u_{\epsilon}) = 0 \]

\[ u_{\epsilon} |_{\partial X} = g \]

\[ M = \sigma(1 + \epsilon c) \frac{\partial u_{\epsilon}}{\partial n} |_{\partial X} \]

[KK '10, BMBT '11]

[Ammari et al. '08]
OT and EIT have high contrast but poor resolution,
pressure waves provide good resolution
combination of the two should be promising.
1 Past work

2 Inverse Diffusion with power densities - Resolution
   - Local reconstruction
   - Global reconstruction
Recalls on the old boundary value problem

On the inversion of the dirichlet-to-neumann map:

- Uniqueness for smooth $\sigma$ [Calderón '80, Sylvester-Uhlmann '87]
- Inversion is severely ill-posed. Logarithmic stability: errors on reconstructions are bounded by $Ce^{1/k}$, where $k$ is the detail size one wants to recovers.
- Consequence: very poor resolution. Practical applications reduced to mere anomaly detection. Still used for its high contrast.
Past work on the inversion

Work on power density measurements:

- Still open: study of the 0-Laplacian: (numerics on it: [Ammari et al '08])

\[ \nabla \cdot \left( H \frac{\nabla u}{|\nabla u|^2} \right) = 0, \quad u|_{\partial X} = g. \]


- Kuchment, Kunyanski '10: synthesized focusing, resolution using small perturbation model + numerics, 2D, 3D.
Outline

1 Past work

2 Inverse Diffusion with power densities - Resolution
   - Local reconstruction
   - Global reconstruction
The problem

Problem: Recover the conductivity $\sigma \geq c_0 > 0$ from the knowledge of $H_{ij}(x) = \sigma \nabla u_i \cdot \nabla u_j$, where $u_i$ solves

$$\nabla \cdot (\sigma \nabla u_i) = 0, \quad u_i \vert_{\partial \Omega} = g_i, \quad 1 \leq i \leq 3.$$

A crucial assumption:

$$\inf_{\Omega} (\det(\nabla u_1, \nabla u_2, \nabla u_3)) \geq c_1 > 0.$$

Set $S_i := \sqrt{\sigma} \nabla u_i$, and $F := \frac{1}{2} \nabla \log \sigma$. The $S_i$'s satisfy the system

$$\nabla \cdot S_i = -F \cdot S_i, \quad \nabla \times S_i = F \times S_i,$$

and the data takes the form $H_{ij} = S_i \cdot S_j$.

Inversion for $\sigma$ requires inverting for $S$ and $F$. 
One can construct an everywhere oriented, orthonormal (i.e. $R^T R = I$, $\det R = 1$) frame $R_i = \sum_{j=1}^{3} t_{ij} S_j$, $1 \leq i \leq 3$, e.g. by Gram-Schmidt. The $R$ frame satisfies

$$\nabla \cdot R_i = -F \cdot R_i + V_{ik} \cdot R_k,$$
$$\nabla \times R_i = F \times R_i + V_{ik} \times R_k, \quad 1 \leq i \leq 3,$$

where the $t_{ij}$’s and $V_{ij}$’s depend on the data.

Overdetermined system: 12 scalar equations for 6 unknown scalar functions.
Resolution. Elimination of the source term

Since $R_i \times R_j = R_k$ for $(1, 2, 3) \rightarrow (i, j, k)$ a direct permutation, and we have the calculus identity

$$\nabla \cdot R_k = \nabla \cdot (R_i \times R_j) = R_j \cdot \nabla \times R_i - R_i \cdot \nabla \times R_j,$$

using these relations together with the $R$ system allows us to derive an equation for $F$

$$F = \frac{1}{n} \left( \frac{1}{2} \nabla \log \det H + \sum_{1 \leq i, j \leq n} ((V_{ij} + V_{ji}) \cdot R_i) R_j \right), \quad (n = 3)$$

Allows us to close the system for the $R_i$'s.
Resolution. First-order ODEs for $R$

Using more vector calculus allows us to derive a system of the form

\[
\begin{align*}
\frac{\partial R}{\partial x_k} &= \sum_{|\alpha| \leq 3} Q_{\alpha,k} R^\alpha, \quad 1 \leq k \leq 3,
\end{align*}
\]

where the functions $Q_{\alpha,k}$ only depend on the data. Still overdetermined, requires integrability conditions.

Reconstruction scheme:

- ODE solving for the $R_i$’s, then one knows $F$.
- reconstruct $\sigma$ by integrating $\nabla \log \sigma = 2F$ along segments.
Choosing appropriate boundary inputs

How to choose boundary conditions so as to guarantee the condition \( \det(\nabla u_1, \nabla u_2, \nabla u_3) = 0 \)?

- 2D: [Alessandrini, Nesi] the BCs \( g_i = x_i, \ i = 1, 2 \) work.
- 3D: [Briane, Milton and Nesi, 04'] there exist conductivities in 3D for which the determinant of the solutions changes sign even with the boundary conditions \( g_i = x_i, 1 \leq i \leq 3 \).
Global reconstruction

Solution: use 4 solutions that alternatively satisfy

\[
\det(S_1, S_2, S_3) \geq c_0 > 0 \quad \text{on } \Omega_{2i},
\]

\[
\det(S_1, S_2, S_4) \geq c_0 > 0 \quad \text{on } \Omega_{2i+1},
\]

for \(1 \leq i \leq N\).
Global reconstruction

Construction of the solutions: uses the Complex Geometric Optics solutions first introduced by Calderón. Assuming $\sigma \in H^{3/2} + \varepsilon$, one can build solutions of $\nabla \cdot (\sigma \nabla u) = 0$ of the form

$$u_r = \sigma^{-\frac{1}{2}} e^{r(k + ik\perp) \cdot x} (1 + \psi_r(x, k)),$$

$$k, k\perp \in \mathbb{R}^3, k \cdot k\perp = 0, \|\psi_r(x)\|_{\mathcal{C}^1} \leq \frac{C_r}{r}.$$ 

oscillates along $k\perp$ and grows exponentially along $k$. For $r$ large enough, we have

$$\Re(\sqrt{\sigma} \nabla u_r) \approx (k \cos(k\perp \cdot x) - k\perp \sin(k\perp \cdot x))e^{k \cdot x},$$

$$\Im(\sqrt{\sigma} \nabla u_r) \approx (k\perp \cos(k\perp \cdot x) + k \sin(k\perp \cdot x))e^{k \cdot x}.$$ 

One does not know $\psi_\rho$ thus use approximate traces to define the $g_i$'s.
Global reconstruction

More generally, one is able to deal with any situation of the form:
Inverse diffusion from power densities

Inverse Diffusion with power densities - Resolution

Global reconstruction

Stability

Theorem (Bal, M., Bonnetier, Triki 11’)

\[ \| \log \sigma - \log \tilde{\sigma} \|_{W^{1,\infty}(X)} \leq C \left( \varepsilon_0 + \| H - \tilde{H} \|_{W^{1,\infty}(X)} \right), \]

where \( \varepsilon_0 \) is the error at the initial point \( x_0 \)

\[ \varepsilon_0 = | \log \sigma_0 - \log \tilde{\sigma}_0 | + \sum_{i=1}^{4} \| S_i(x_0) - \tilde{S}_i(x_0) \|. \]

This inversion is well-posed, i.e. resolution on \( \sigma \) is as good as that on the data \( H \).
Conclusion

Ultrasound modulations in conjunction with high-contrast techniques offer promising improvements over regular OT and EIT:

- explicit reconstruction formula,
- well-posed inverse problems, therefore good resolution,
- local (i.e. requires partial knowledge at the boundary).

Compatible with the fact that we cannot necessarily ensure 3 everywhere linearly independent solutions.

Future work:

- generalization to dimension $n$,
- numerical implementations in 3D,
- accounting for noise, characterizing the range of the measurement operator.