Coupled-physics Inverse Problems for the System of Elasticity

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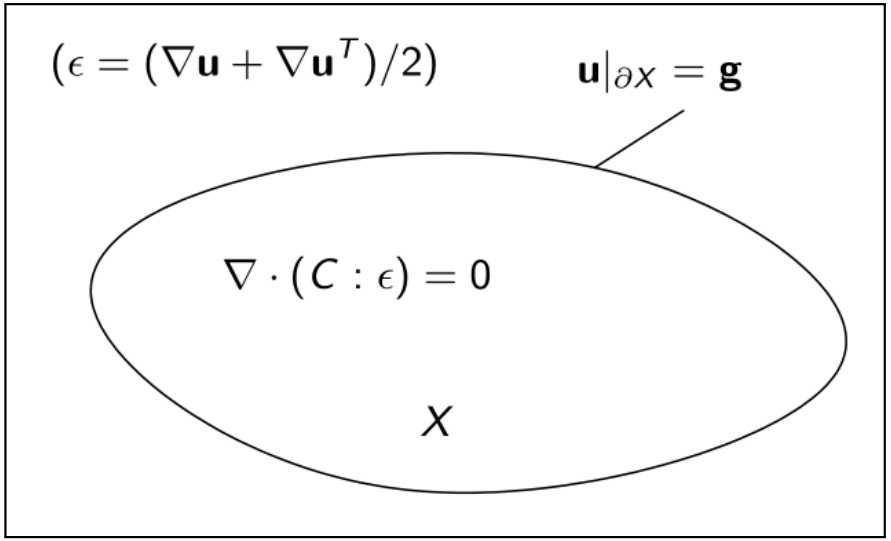
Joint works with Guillaume Bal, Cédric Bellis, Sébastien Imperiale and Gunther Uhlmann

Outline

Inverse elasticity from internal measurements

The model: $X \subset \mathbb{R}^3$ bounded domain.



Physical assumptions:

C 4th-order tensor with symmetries

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$$

Uniform ellipticity:

Exists $\kappa > 0$ such that

$$\epsilon : C(x) : \epsilon \geq \kappa \epsilon : \epsilon,$$

for all $\epsilon \in S_3(\mathbb{R})$ and $x \in X$.

Internal measurements:

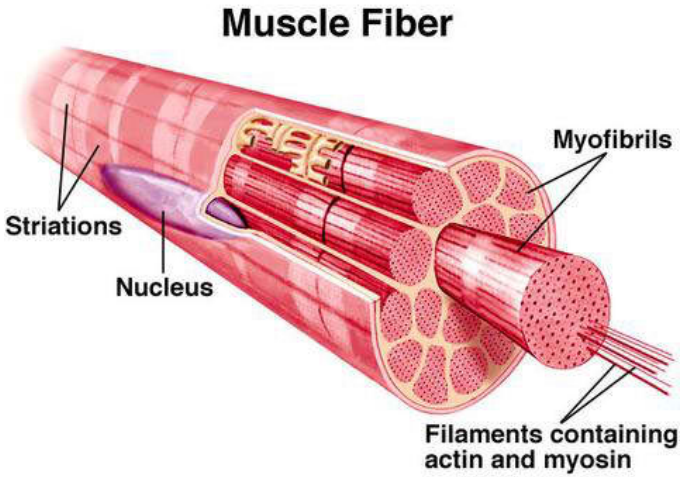
- $\mathbf{u}$ inside the domain

Inverse problem considered:

Does a finite collection of $\mathbf{u}$'s determine C

- uniquely ?
- stably ?

Anisotropy classes



Transversely anisotropic material

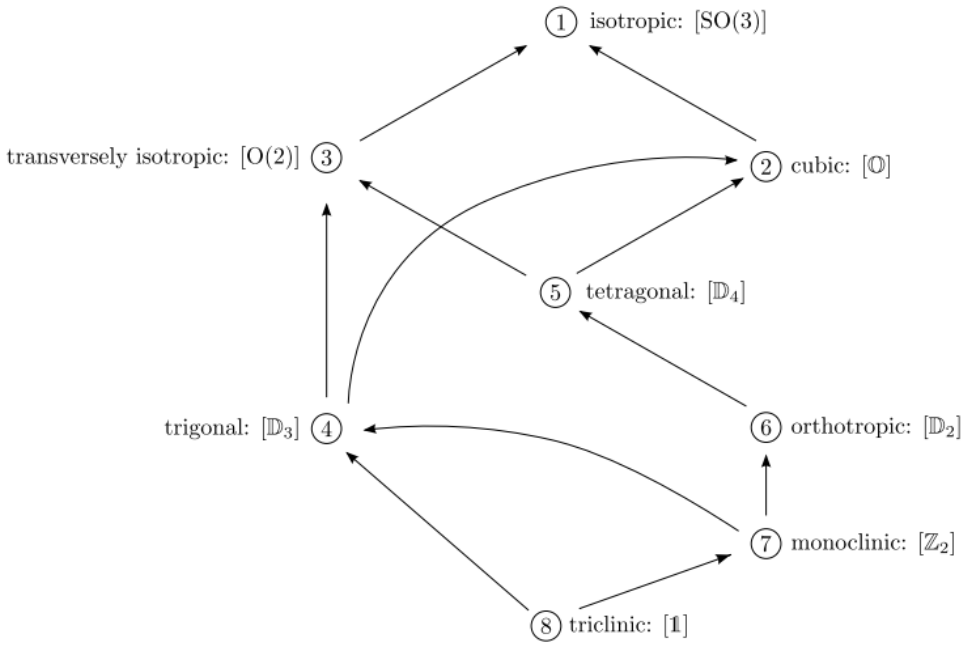


FIGURE 1. The lattice of isotropy classes for $\mathbb{H}^4$.

[Auffray et al., *Invariant based approach to symmetry class detection*]

Litterature on hybrid inversions

Scalar PDEs

- Inverse conductivity from power/current densities, general elliptic PDEs: Ammari, Capdeboscq, Kuchment, Kunyansky, Monard, Bal, Bonnetier, Triki, Guo, Steinhauer, Gebauer-Scherzer, Nachman, Tamasan, Timonov, Uhlmann, Hoell, Moradifam, Kocyigit, Montalto, Stefanov, Alberti (2min notice) ...

Elasticity: (today's talk)

- shear modulus: [Barbone et al., '10] [McLaughlin et al., '10]
- Lamé system: [Bal-Bellis-Imperiale-M., '14] [Lai, '14]
- General anisotropic: [Bal-M.-Uhlmann, preprint]

Other systems: Maxwell

- [Bal-Guo, '14], [Bal-Zhou, '14]

General approach with internal measurements

A tradeoff from experience on hybrid problems for scalar PDEs:

Minimal measurements versus maximal stability and locality of inversion (e.g. inverse conductivity from power/current densities: p-Laplacians versus pointwise algebraic formulas).

The approach:

1. Considering "many" measurements, exploit linear algebraic redundancies. Find sufficient conditions for most stable inversion.
2. Classify tensors for which these conditions can be satisfied.

Outline

The problem

Problem: In the particular case $C(\lambda, \mu) = \lambda \mathbb{I}_2 \otimes \mathbb{I}_2 + 2\beta \mathbb{I}_4$, reconstruct the Lamé parameters (λ, μ) from knowledge of solutions $\mathbf{u}$ to the equation

$$\nabla(\lambda \nabla \cdot \mathbf{u}) + \nabla \cdot (2\mu\epsilon) + \rho\omega^2 \mathbf{u} = 0 \quad (X), \qquad \mathbf{u}|_{\partial X} = \mathbf{g} \quad \text{(prescribed)},$$

where $\epsilon = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$.

Dimension analysis: two solutions $\mathbf{u}_1, \mathbf{u}_2$ should be enough to obtain a gradient system for (λ, μ).

Main hypothesis: Define $t_i = \mathrm{tr}(\epsilon_i)$ and $\epsilon_i^D = \epsilon_i - \frac{1}{3} t_i \mathbb{I}_3$ $(i = 1, 2)$.

Hyp: The tensor $\mathcal{E} = t_1 \epsilon_2^D - t_2 \epsilon_1^D$ is uniformly invertible throughout X.

Inversion

By elimination, and assuming the hypothesis above, one can derive a system of the form

$$\nabla\lambda + A_{11}\lambda + A_{12}\mu = B_{11}\mathbf{u}_1 + B_{12}\mathbf{u}_2,$$
$$\nabla\mu + A_{21}\lambda + A_{22}\mu = B_{21}\mathbf{u}_1 + B_{22}\mathbf{u}_2,$$

where $A_{ij}, B_{ij} : X \to \mathbb{R}^3$ are functions of ϵ_1, ϵ_2, ω (known).

Possible inversion approaches:

- By integration along chosen curves.
- By variational formulation. ([Barbone et al., '10])

Injectivity/stability

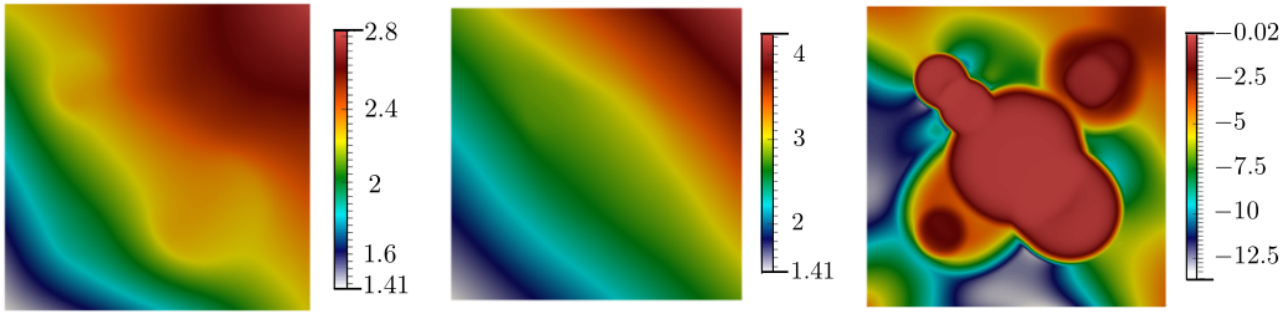
Using Gronwall's lemma, the ODE integration method yields a stability result of the form

$$\|\lambda - \lambda'\|_{W^{p+1,\infty}(\Omega)} + \|\mu - \mu'\|_{W^{p+1,\infty}(\Omega)} \leq C\big(\epsilon_0 + \sum_{i=1}^{2} \|\mathbf{u}_i - \mathbf{u}_i'\|_{W^{p+2,\infty}(\Omega)}\big)$$

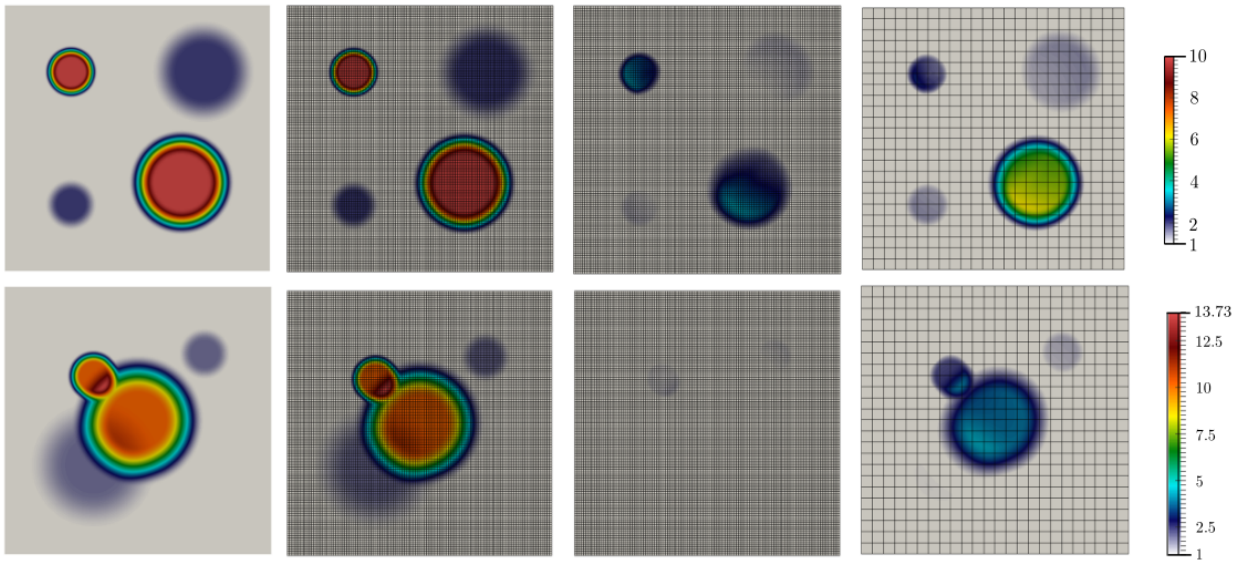
for $\Omega \subseteq X$ where the hypothesis is satisfied. Loss of one derivative.

Similar estimates in Sobolev norms for the variational approach.

Numerical examples (frequency-dependent)



(a) $|\boldsymbol{u}_1(\boldsymbol{x})|$ (b) $|\boldsymbol{u}_2(\boldsymbol{x})|$ (c) $\det \mathscr{E}(\boldsymbol{x})$



(a) $(\alpha(\boldsymbol{x}), \beta(\boldsymbol{x}))$ (b) $\delta = 0, k = 1$ (c) $\delta = 10^{-7}, k = 1$ (d) $\delta = 10^{-7}, k = 5$

Figure 6: (a) Exact values of $\alpha(\boldsymbol{x})$ *(top)* and $\beta(\boldsymbol{x})$ *(bottom)*; Corresponding reconstructions with (b) no noise nor regularization, (c) with noise but no regularization, (d) with noise and regularization.

Outline

The problem

Problem: recover C from knowledge of $\{\mathbf{u}^{(j)}\}_{j=1}^{6+N}$, where $\mathbf{u}^{(j)}$ solves

$$\nabla \cdot (C : \epsilon) = 0 \quad (X), \quad \mathbf{u}|_{\partial X} = \mathbf{g} \quad \text{(prescribed)},$$

where $\epsilon = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$.

Two crucial assumptions: The family $\{\mathbf{u}^{(j)}\}_{j=1}^{6+N}$ satisfies

$$(A) \quad \inf_X \det(\epsilon^{(1)}, \ldots, \epsilon^{(6)}) \geq c_0 > 0$$

$$(B) \quad \text{span}(M_1, \ldots M_{3N}) \text{ has codimension one in } S_6(\mathbb{R}),$$

where the matrices M_j are constructed in terms of $\{\epsilon^{(j)}\}_{k=1}^{6+N}$ and their first derivatives, i.e., known from data.

Inversion strategy (1/3)

Rewrite the problem in non-tensorial notation:

$C \leftrightarrow \mathbf{c}$ in Voigt notation via the double index mapping

$$11 \mapsto,\ 22 \mapsto 2,\ 33 \mapsto 3,\ 23, 32 \mapsto 4,\ 13, 31 \mapsto 5,\ 12, 21 \mapsto 6,$$

e.g. $\mathbf{c}_1 = C_{1111}$, etc... Then $\mathbf{c} \in S_6(\mathbb{R})$. Define

$$D_V = \begin{bmatrix} \partial_1 & 0 & 0 & 0 & \partial_3 & \partial_2 \\ 0 & \partial_2 & 0 & \partial_3 & 0 & \partial_1 \\ 0 & 0 & \partial_3 & \partial_2 & \partial_1 & 0 \end{bmatrix},$$

then the system of elastostatics reads

$$\boxed{D_V \cdot (\mathbf{c}\ \epsilon) = 0}, \qquad \epsilon := (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{23}, 2\epsilon_{31}, 2\epsilon_{12})^T.$$

Inversion strategy (2/3)

Assumption (A): $\epsilon^{(1)}, \ldots, \epsilon^{(6)}$ basis of $\mathbb{R}^6$ at every $x \in X$, thus for $1 \leq p \leq N$, $\epsilon^{(6+p)}$ may decompose in this basis in the form

$$\epsilon^{(6+p)} = \mu_{p1}\epsilon^{(1)} + \cdots + \mu_{p6}\epsilon^{(6)},$$

where μ_{pj} are known from data. Applying $D_V(\mathbf{c}\cdot)$ to this relation, we obtain

$$\boxed{\sum_{j=1}^{6} (D_V \mu_{pj}) \cdot \mathbf{c} \cdot \epsilon^{(j)} = 0}$$

(3 orthogonality constraints on $\mathbf{c}$ in $S_6(\mathbb{R})$)

If enough linearly independent constraints are achieved at some $x \in X$ (Assumption (B)), $\mathbf{c}(x)$ can be reconstructed up to a scalar constant via a generalization of cross-product.

<u>General:</u> 21 unknowns $\rightarrow$ 20 constraints $\rightarrow$ need $6 + 7$ solutions.

<u>Trans. Anis.:</u> 5 unknowns $\rightarrow$ 4 constraints $\rightarrow$ need $6 + 2$ solutions.

Inversion strategy (3/3)

Assume $\tilde{\mathbf{c}}$ such that $\det \tilde{\mathbf{c}} = 1$ is reconstructed.

Reconstruct missing scalar factor: write $C = \tau \widetilde{C}$ where $\widetilde{C} \leftrightarrow \tilde{\mathbf{c}}$.

The system of elasticity reads

$$(\widetilde{C} : \epsilon)\nabla \log \tau = -\mathrm{div}(\widetilde{C} : \epsilon).$$

Using assumption (A) and writing $\mathbb{I}_3 = \sum_{j=1}^{6} \mu_j \epsilon^{(j)}$, we get

$$\nabla \log \tau = -\sum_{j=1}^{6} \mu_j \ \mathrm{div}(\widetilde{C} : \epsilon^{(j)}).$$

Gain stability on $\mathrm{div} C := \partial_i C_{ijkl} \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$: From

$$(\mathrm{div} C)_{j..} : \epsilon = -C_{ijkl} \partial_i \epsilon_{kl}, \qquad 1 \leq j \leq 3,$$

- can reconstruct $\mathrm{div} C$ with the same stability as that on C.

Injectivity/stability

Theorem (Bal-M.-Uhlmann '15)

If $\{\mathbf{u}^{(j)}\}_{j=1}^{6+N}$ satisfy the reconstructibility conditions (A) and (B) throughout $\Omega \subseteq X$, then C is uniquely determined on Ω up to a scalar constant, with a stability of the form:

$$\|C - C'\|_{W^{p,\infty}(\Omega)} + \|divC - divC'\|_{W^{p,\infty}(\Omega)} \leq K \sum_{j=1}^{N+6} \|\epsilon^{(j)} - \epsilon'^{(j)}\|_{W^{p+1,\infty}(\Omega)},$$

whenever $\Omega \subseteq X$ and $\{\mathbf{u}^{(j)}\}_{j=1}^{6+N}$, $\{\mathbf{u}'^{(j)}\}_{j=1}^{6+N}$ both satisfy (A) and (B) throughout Ω.

- loss of two derivatives on C, but also on functionals of ∇C.
- sharp.

What boundary conditions work ? 1/2

Question: How do we find $\mathbf{g}_1, \ldots, \mathbf{g}_{6+N}$ such that

(A) $\inf_X \det(\epsilon^{(1)}, \ldots, \epsilon^{(6)}) \geq c_0 > 0$?

(B) $\mathrm{span}(M_1, \ldots, M_{3N})$ has codimension one ?

First case: Constant and near-constant tensors

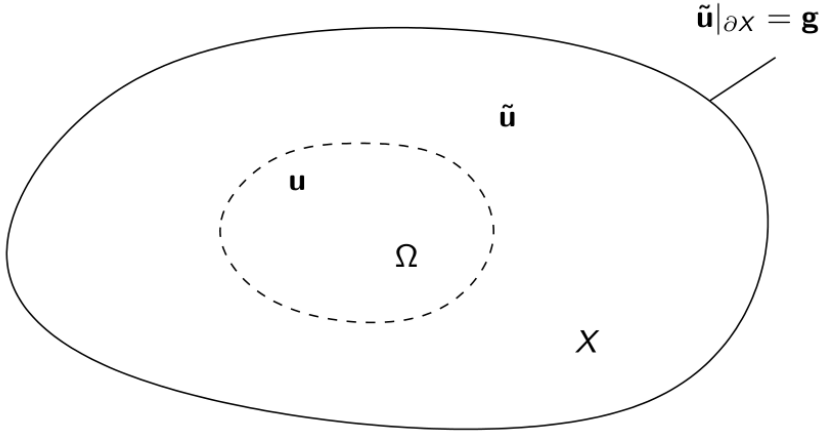
- Constant tensor: can achieve (A) and (B) via explicit displacement fields with linear and quadratic components.
- Tensor $\mathcal{C}^3$-close to constant: perturbation arguments.

Second case: Runge approximation, as in [Bal-Uhlmann, CPAM '13].

- Only holds so far for isotropic and transversely isotropic tensors. [Nakamura-Uhlmann-Wang, '05]
- When it holds, reconstructibility is proved by constructing local solutions with appropriate behavior.

What boundary conditions work ? 2/2

Runge: $\forall \varepsilon > 0, \forall \mathbf{u}, \nabla \cdot (C : \epsilon) = 0, \quad \exists \tilde{\mathbf{u}}, \quad \|\mathbf{u} - \tilde{\mathbf{u}}\|_{L(\Omega)} \leq \varepsilon$



Ideas:

- Freeze coefficients on a small ball, construct explicit solutions satisfying hypotheses (A)-(B).
- For C smooth enough, L^2 approximation yields $\mathcal{C}^2$ approximation on smaller sets.
- Control these solutions from the boundary via Runge $\mathcal{C}^2$-approximation.

Conclusion

Measurements considered provide

- Full inversion formulas under knowledge of enough displacement fields,
- For Lamé system: Hölder stability for (λ, μ), loss of one derivative.
- For the anisotropic case:
  - Full reconstruction in the most general case.
  - Hölder stability (loss of two derivatives) for C and $\mathrm{div} C$.

Future directions:

- Add time-harmonic term $\rho\omega^2 \mathbf{u}$ to the anisotropic case.
- Prove more Runge approximations theorems or find more explicit ways to fulfill the reconstructibility conditions.
- Numerics in 3D.

Thank you !

• G. Bal, C. Bellis, S. Imperiale and F.M., *Reconstruction of constitutive parameters in isotropic linear elasticity from noisy full-field measurements*, Inverse Problems 30 (2014) 125004

• G. Bal, F.M. and G. Uhlmann, *Reconstruction of a fully anisotropic elasticity tensor from knowledge of displacement fields*, submitted (2015). arXiv:1501.05361

References and slides available at
`http://www.math.washington.edu/~fmonard/research.html`