Coupled-physics Inverse Problems for the System of Elasticity

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Joint works with Guillaume Bal, Cédric Bellis, Sébastien Imperiale and Gunther Uhlmann
Inverse elasticity from internal measurements

The model: $X \subset \mathbb{R}^3$ bounded domain.

Uniform ellipticity:
Exists $\kappa > 0$ such that

$$\epsilon : C(x) : \epsilon \geq \kappa \epsilon : \epsilon,$$

for all $\epsilon \in S_3(\mathbb{R})$ and $x \in X$.

Internal measurements:
- $u$ inside the domain

Inverse problem considered:
Does a finite collection of $u$’s determine $C$
- uniquely?
- stably?

Physical assumptions:
$C$ $4^{th}$-order tensor with symmetries

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$$
Anisotropy classes

Transversely anisotropic material

[Figure 1. The lattice of isotropy classes for $H^4$.]

[Auffray et al., Invariant based approach to symmetry class detection]
Litterature on hybrid inversions

Scalar PDEs

- Inverse conductivity from power/current densities, general elliptic PDEs: Ammari, Capdeboscq, Kuchment, Kunyansky, Monard, Bal, Bonnetier, Triki, Guo, Steinhauer, Gebauer-Scherzer, Nachman, Tamasan, Timonov, Uhlmann, Hoell, Moradifam, Kocyigit, Montalto, Stefanov, Alberti (2min notice) ...

Elasticity: (today’s talk)

- shear modulus: [Barbone et al., ’10] [McLaughlin et al., ’10]
- Lamé system: [Bal-Bellis-Imperiale-M., ’14] [Lai, ’14]
- General anisotropic: [Bal-M.-Uhlmann, preprint]

Other systems: Maxwell

- [Bal-Guo, ’14], [Bal-Zhou, ’14]
General approach with internal measurements

A tradeoff from experience on hybrid problems for scalar PDEs: Minimal measurements versus maximal stability and locality of inversion (e.g. inverse conductivity from power/current densities: $p$-Laplacians versus pointwise algebraic formulas).

The approach:

2. Classify tensors for which these conditions can be satisfied.
1 Introduction

2 The Lamé system

3 General anisotropic case
The problem

Problem: In the particular case \( C(\lambda, \mu) = \lambda \mathbb{I}_2 \otimes \mathbb{I}_2 + 2\beta \mathbb{I}_4 \), reconstruct the Lamé parameters \((\lambda, \mu)\) from knowledge of solutions \(u\) to the equation

\[
\nabla(\lambda \nabla \cdot u) + \nabla \cdot (2\mu \epsilon) + \rho \omega^2 u = 0 \quad (X), \quad u|_{\partial X} = g \quad \text{(prescribed)},
\]

where \(\epsilon = \frac{1}{2}(\nabla u + \nabla u^T)\).

Dimension analysis: two solutions \(u_1, u_2\) should be enough to obtain a gradient system for \((\lambda, \mu)\).

Main hypothesis: Define \(t_i = \text{tr}(\epsilon_i)\) and \(\epsilon_i^D = \epsilon_i - \frac{1}{3} t_i \mathbb{I}_3\) \((i = 1, 2)\).

Hyp: The tensor \(E = t_1 \epsilon_2^D - t_2 \epsilon_1^D\) is uniformly invertible throughout \(X\).
By elimination, and assuming the hypothesis above, one can derive a system of the form

\[ \nabla \lambda + A_{11} \lambda + A_{12} \mu = B_{11} u_1 + B_{12} u_2, \]
\[ \nabla \mu + A_{21} \lambda + A_{22} \mu = B_{21} u_1 + B_{22} u_2, \]

where \( A_{ij}, B_{ij} : X \to \mathbb{R}^3 \) are functions of \( \epsilon_1, \epsilon_2, \omega \) (known).

Possible inversion approaches:

- By integration along chosen curves.
- By variational formulation. ([Barbone et al., ’10])
Injectivity/stability

Using Gronwall’s lemma, the ODE integration method yields a stability result of the form

$$\|\lambda - \lambda'\|_{W^{p+1,\infty}(\Omega)} + \|\mu - \mu'\|_{W^{p+1,\infty}(\Omega)} \leq C(\epsilon_0 + \sum_{i=1}^{2} \|u_i - u_i'\|_{W^{p+2,\infty}(\Omega)})$$

for $\Omega \subseteq \mathcal{X}$ where the hypothesis is satisfied. Loss of one derivative. Similar estimates in Sobolev norms for the variational approach.
Numerical examples (frequency-dependent)

Figure 6: (a) Exact values of $\alpha(\mathbf{x})$ (top) and $\beta(\mathbf{x})$ (bottom); Corresponding reconstructions with (b) no noise nor regularization, (c) with noise but no regularization, (d) with noise and regularization.
Outline

1 Introduction

2 The Lamé system

3 General anisotropic case
The problem

Problem: recover $C$ from knowledge of $\{u^{(j)}\}_{j=1}^{6+N}$, where $u^{(j)}$ solves

$$\nabla \cdot (C : \epsilon) = 0 \quad (X), \quad u|_{\partial X} = g \quad \text{(prescribed)},$$

where $\epsilon = \frac{1}{2}(\nabla u + \nabla u^T)$.

Two crucial assumptions: The family $\{u^{(j)}\}_{j=1}^{6+N}$ satisfies

(A) $\inf_X \det(\epsilon^{(1)}, \ldots, \epsilon^{(6)}) \geq c_0 > 0$

(B) $\text{span}(M_1, \ldots, M_{3N})$ has codimension one in $S_6(\mathbb{R})$,

where the matrices $M_j$ are constructed in terms of $\{\epsilon^{(j)}\}_{j=1}^{6+N}$ and their first derivatives, i.e., known from data.
Inversion strategy (1/3)

Rewrite the problem in non-tensorial notation:

\[ C \leftrightarrow c \] in Voigt notation via the double index mapping

\[ 11 \mapsto 1, \ 22 \mapsto 2, \ 33 \mapsto 3, \ 23, 32 \mapsto 4, \ 13, 31 \mapsto 5, \ 12, 21 \mapsto 6, \]

e.g. \( c_1 = C_{1111} \), etc... Then \( c \in S_6(\mathbb{R}) \). Define

\[
D_V = \begin{bmatrix}
\partial_1 & 0 & 0 & 0 & \partial_3 & \partial_2 \\
0 & \partial_2 & 0 & \partial_3 & 0 & \partial_1 \\
0 & 0 & \partial_3 & \partial_2 & \partial_1 & 0
\end{bmatrix},
\]

then the system of elastostatics reads

\[
D_V \cdot (c \ \epsilon) = 0, \quad \epsilon := (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{23}, 2\epsilon_{31}, 2\epsilon_{12})^T.
\]
Inversion strategy (2/3)

Assumption (A): \( \epsilon^{(1)}, \ldots, \epsilon^{(6)} \) basis of \( \mathbb{R}^6 \) at every \( x \in X \), thus for \( 1 \leq p \leq N \), \( \epsilon^{(6+p)} \) may decompose in this basis in the form

\[
\epsilon^{(6+p)} = \mu_{p1}\epsilon^{(1)} + \cdots + \mu_{p6}\epsilon^{(6)},
\]

where \( \mu_{pj} \) are known from data. Applying \( DV(c \cdot) \) to this relation, we obtain

\[
\sum_{j=1}^{6} (DV\mu_{pj}) \cdot c \cdot \epsilon^{(j)} = 0 \quad (3 \text{ orthogonality constraints on } c \text{ in } S_6(\mathbb{R}))
\]

If enough linearly independent constraints are achieved at some \( x \in X \) (Assumption (B)), \( c(x) \) can be reconstructed up to a scalar constant via a generalization of cross-product.

General: 21 unknowns \( \rightarrow \) 20 constraints \( \rightarrow \) need 6 + 7 solutions.

Trans. Anis.: 5 unknowns \( \rightarrow \) 4 constraints \( \rightarrow \) need 6 + 2 solutions.
Inversion strategy (3/3)

Assume \( \tilde{c} \) such that \( \det \tilde{c} = 1 \) is reconstructed.

Reconstruct missing scalar factor: write \( C = \tau \tilde{C} \) where \( \tilde{C} \leftrightarrow \tilde{c} \).

The system of elasticity reads

\[
(\tilde{C} : \epsilon) \nabla \log \tau = -\text{div}(\tilde{C} : \epsilon).
\]

Using assumption (A) and writing \( \mathbb{I}_3 = \sum_{j=1}^{6} \mu_j \epsilon^{(j)} \), we get

\[
\nabla \log \tau = -\sum_{j=1}^{6} \mu_j \text{div}(\tilde{C} : \epsilon^{(j)}).
\]

Gain stability on \( \text{div}C := \partial_i C_{ijkl} e_j \otimes e_k \otimes e_l \):

From

\[
(\text{div}C)_{j..} : \epsilon = -C_{ijkl} \partial_i \epsilon_{kl}, \quad 1 \leq j \leq 3,
\]

\( \triangleright \) can reconstruct \( \text{div}C \) with the same stability as that on \( C \).
Theorem (Bal-M.-Uhlmann ’15)

If \( \{u^{(j)}\}_{j=1}^{6+N} \) satisfy the reconstructibility conditions (A) and (B) throughout \( \Omega \subseteq X \), then \( C \) is uniquely determined on \( \Omega \) up to a scalar constant, with a stability of the form:

\[
\| C - C' \|_{W^{p,\infty}(\Omega)} + \| \text{div} C - \text{div} C' \|_{W^{p,\infty}(\Omega)} \leq K \sum_{j=1}^{N+6} \| \epsilon^{(j)} - \epsilon'(j) \|_{W^{p+1,\infty}(\Omega)},
\]

whenever \( \Omega \subseteq X \) and \( \{u^{(j)}\}_{j=1}^{6+N}, \{u'(j)\}_{j=1}^{6+N} \) both satisfy (A) and (B) throughout \( \Omega \).

- loss of two derivatives on \( C \), but also on functionals of \( \nabla C \).
- sharp.
What boundary conditions work? 1/2

Question: How do we find $g_1, \ldots, g_{6+N}$ such that

(A) $\inf_X \det(\epsilon^{(1)}, \ldots, \epsilon^{(6)}) \geq c_0 > 0$ ?

(B) $\text{span}(M_1, \ldots, M_{3N})$ has codimension one ?

First case: Constant and near-constant tensors

- Constant tensor: can achieve (A) and (B) via explicit displacement fields with linear and quadratic components.
- Tensor $C^3$-close to constant: perturbation arguments.

Second case: Runge approximation, as in [Bal-Uhlmann, CPAM ’13].

- Only holds so far for isotropic and transversely isotropic tensors. [Nakamura-Uhlmann-Wang, ’05]
- When it holds, reconstructibility is proved by constructing local solutions with appropriate behavior.
What boundary conditions work? 2/2

Runge: $\forall \varepsilon > 0, \forall u, \nabla \cdot (C : \varepsilon) = 0, \exists \tilde{u}, \|u - \tilde{u}\|_{L(\Omega)} \leq \varepsilon$

Ideas:
- Freeze coefficients on a small ball, construct explicit solutions satisfying hypotheses (A)-(B).
- For $C$ smooth enough, $L^2$ approximation yields $C^2$ approximation on smaller sets.
- Control these solutions from the boundary via Runge $C^2$-approximation.
Conclusion

Measurements considered provide

- Full inversion formulas under knowledge of enough displacement fields,
- For Lamé system: Hölder stability for $(\lambda, \mu)$, loss of one derivative.
- For the anisotropic case:
  - Full reconstruction in the most general case.
  - Hölder stability (loss of two derivatives) for $C$ and $\text{div} C$.

Future directions:

- Add time-harmonic term $\rho \omega^2 u$ to the anisotropic case.
- Prove more Runge approximations theorems or find more explicit ways to fulfill the reconstructibility conditions.
- Numerics in 3D.
Thank you!


References and slides available at
http://www.math.washington.edu/~fmonard/research.html