Coupled-physics methods for inverse conductivity

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Outline

1 introduction

2 Inversion from power densities
   - Reconstruction of the scalar factor
   - Reconstruction of the anisotropic structure

3 Inversion from current densities
   - Reconstruction of the scalar factor
   - Reconstruction of the anisotropic structure
Coupled-physics methods for inverse conductivity

**Introduction**

Inverse conductivity from internal measurements

**The model:** $X \subset \mathbb{R}^n$ bounded domain.

$\gamma$ is uniformly elliptic.

**Internal measuresnts:**

- **Power density operator:**
  \[ \mathcal{H}_\gamma(g) = \gamma \nabla u \cdot \nabla u \]

- **Current density operator:**
  \[ \mathcal{J}_\gamma(g) = \gamma \nabla u \]

**Inverse problems considered:**

Does $\mathcal{H}_\gamma / \mathcal{J}_\gamma$ determine $\gamma$ uniquely? stably?
Derivation of power densities - 1/2

By ultrasound modulation

Physical focusing

[Ammari et al. '08]

Synthetic focusing

[Kuchment-Kunyansky '10]
[Bal-Bonnetier-M.-Triki '11]

Small perturbation model:

\[ \frac{(\mathcal{M}_\epsilon - \mathcal{M}_0)}{\epsilon} \] gives an approximation of \( \nabla u_0 \cdot \gamma \nabla u_0 \) at \( x_0 \).
Derivation of power densities - 2/2

By thermoelastic effects (Impedance-Acoustic CT)

1: voltage is prescribed at $\partial X$
   \[ u|_{\partial X} = g(x)\delta(t) \]

2: currents are generated inside the domain
   \[ \nabla \cdot (\gamma \nabla u) = 0 \]

3: the energy absorbed generates elastic waves
   \[ \frac{1}{v_s^2} \frac{\partial^2 p}{\partial t^2} - \Delta p = 0 \]
   \[ p|_{t=0} = \Gamma H_\gamma[g] \]
   \[ \partial_t p|_{t=0} = 0 \]

4: waves are measured at $\partial X$ by ultrasound transducers

One reconstructs $\Gamma H_\gamma = \Gamma \nabla u \cdot \gamma \nabla u$ over $X$ ($\Gamma$: Grüneisen coefficient)

[Gebauer-Scherzer ’09]
Derivation of current densities

Context: Current Density Imaging (CDI) or Magnetic Resonance Electrical Impedance Tomography (MREIT): coupling conductivity (high contrast) with magnetic imaging (high resolution).

- MRI measurements allow to measure the magnetic field $B$.
- The current density is then obtained as $J = \frac{1}{\mu_0} \nabla \times B$.

More realistic: access to $|J|$ or a component of the vector $J$.

[Scott-Joy-Armstrong-Henkelman ’91]
References on inversion (from many functionals)

Inversion from power densities:
- 2D isotropic: [Capdeboscq et al. '09].
- 2D-3D isotropic linearized: [Kuchment-Kunyansky '11].
- 2D-3D isotropic: [Bal-Bonnetier-M.-Triki, IPI '12].
- nD isotropic: [M.-Bal, IPI '12], [Kocyigit '12].
- 2D anisotropic: [M.-Bal, IP '12].
- nD anisotropic: [M.-Bal, CPDE '13].

Pseudodifferential analysis on linearized power densities:
- Isotropic case: [Kuchment-Steinhauer, '12].
- nD anisotropic case: [Bal-Guo-M., preprint].

Inversion from many current densities:
- nD isotropic: [M.-Bal, IPI '12].
- nD anisotropic case: [Bal-Guo-M., preprint].
References on inversion (from few functionals)

Resolution from a single power density - isotropic case:

- Newton-based numerical methods to recover \((u, \gamma)\)  
  [Ammari et al. '08, Gebauer-Scherzer '09].
- Theoretical work on the Cauchy problem  [Bal '11].

Resolution from a single current density - isotropic case:

- Isotropic:  [Nachman-Tamasan-Timonov '07, '09, '10, '11]
- Anisotropic with known anisotropy:  
  [Hoell-Moradifam-Nachman, preprint]
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Formulation of the problem

**Problem:** Reconstruct \( \beta := (\det \gamma)^{\frac{1}{n}} \) (anisotropy \( \tilde{\gamma} := (\det \gamma)^{-\frac{1}{n}} \gamma \) known) from \( H_{ij}(x) = \gamma \nabla u_i \cdot \nabla u_j \), where \( u_i \) solves

\[
\nabla \cdot (\gamma \nabla u_i) = 0, \quad u_i|_{\partial \Omega} = g_i, \quad 1 \leq i \leq n.
\]

**Assumption:**

- \( \inf_{\Omega} \det(\nabla u_1, \ldots, \nabla u_n) \geq c_1 > 0 \) over some \( \Omega \subset X \).

Define \( A = \gamma^{\frac{1}{2}} = \sqrt{\beta} \tilde{A} \) and \( S_i := A \nabla u_i \). The \( S_i \)'s satisfy, for \( 1 \leq i \leq n \)

\[
\nabla \cdot (\tilde{A} S_i) = -\frac{1}{2} \nabla \log \beta \cdot \tilde{A} S_i, \quad d(\tilde{A}^{-1} S_i) = \frac{1}{2} \nabla \log \beta \wedge \tilde{A}^{-1} S_i.
\]

The data takes the form \( H_{ij} = S_i \cdot S_j \) (Grammian matrix).

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Legend: known data, anisotropic structure, unknown.
Derivation of a local transport equation

Using the equations $\nabla \cdot (AS_i) = 0$ and $d(A^{-1}S_i) = 0$, one is able to derive the transport equation

$$\nabla \log \beta = \frac{1}{n|H|^\frac{1}{2}} \left( \nabla (|H|^\frac{1}{2} H^{ij}) \cdot \tilde{A}S_i \right) \tilde{A}^{-1} S_j.$$ 

A first-order quasi-linear system is then derived for the frame $S$

$$\nabla S_i = S_i(S, \tilde{A}, d\tilde{A}, H, dH), \quad 1 \leq i \leq n$$

where $S_i$ is Lipschitz w.r.t. $(S_1, \ldots, S_n)$.

▷ Overdetermined PDEs, solvable for $S$ over $\Omega \subset X$ via ODE’s along any (characteristic) curves.
Global patching

Patching local ODE-based reconstructions:
Assume $X \subset \bigcap_{i=1}^{N} \mathcal{O}_i$, where for each $\mathcal{O}_i$, there exists $(u_{i_1}, \ldots, u_{i_n}) \subset (u_1, \ldots, u_m)$ satisfying

$$\inf_{\mathcal{O}_i} \det(\nabla u_{i_1}, \ldots, \nabla u_{i_n}) \geq c_0 > 0.$$ 

Over each $\mathcal{O}_i$: solve by ODE integration

$$\nabla \log \beta = \mathcal{F}(S, H, dH, \tilde{A}),$$
$$\nabla S_{i_p} = S_{i_p}(S, H, dH, \tilde{A}, d\tilde{A}),$$
$$1 \leq p \leq n.$$
Improvement over the patching

More direct assumption (without the open cover):

\[ \inf_{x} D(x) \geq c_0 > 0, \quad D := \sum_{1 \leq i_1 < \ldots < i_n \leq m} \det \{ H_{i_p i_q} \}_{1 \leq p, q \leq n}. \]

Merge all the transport equations into a single one whose denominator becomes \( D \) (never vanishing by assumption). Global reconstruction without having to patch

- Either by ODE solving,
- Or by taking divergence and solving a Poisson equation.
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Stability

Theorem (Uniqueness and Lipschitz stability in $W^{1,\infty}(\Omega)$)

Over $\Omega \subset X$ where the frame assumption is satisfied, $|\gamma|$ is uniquely determined up to a constant. Moreover,

$$\| \log |\gamma| - \log |\gamma'| \|_{W^{1,\infty}} \leq \varepsilon_0 + C(\|H - H'\|_{W^{1,\infty}} + \|\tilde{A} - \tilde{A}'\|_{W^{1,\infty}})$$

[Capdeboscq et al. ’09], [Bal-Bonnetier-M.-Triki, ’12], [M.-Bal, IP ’12], [M.-Bal, IPI ’12], [M.-Bal, CPDE ’13]

▷ Well-posed problem if the anisotropy is known.
▷ No loss of derivative/resolution on $|\gamma|$. 
Formulation

**Problem:** reconstruct $\tilde{\gamma}$ from power densities.

**Assumption:** Start with a "basis" $(u_1, \ldots, u_n)$ of solutions, and consider additional solutions with their power densities.

**Important observation:** For an additional solution $u_{n+1}$, we decompose $\nabla u_{n+1}$ as

$$\nabla u_{n+1} = \sum_{i=1}^{n} \mu_i \nabla u_i,$$

$$\mu_j = \frac{\det(\nabla u_1, \ldots, \nabla u_{n+1}, \ldots, \nabla u_n)}{\det(\nabla u_1, \ldots, \nabla u_n)}.$$

The coefficients $\mu_j$ are known from power densities.

Hit the relation above with $\nabla \cdot (\gamma \cdot)$ and $d \cdot$ to make appear

$$\nabla \mu_i \cdot \tilde{\gamma} \nabla u_i = 0,$$

and

$$\nabla \mu_i \wedge \nabla u_i = 0.$$
Algebraic reconstruction algorithm

If $\tilde{A} = \tilde{\gamma}^{\frac{1}{2}}$ and $S = A[\nabla u_1| \ldots |\nabla u_n]$, the previous conditions can be written as (denote $Z := [\nabla \mu_1, \ldots, \nabla \mu_n]$)

$$\tilde{A}S : Z = 0 \quad \text{and} \quad \tilde{A}S : ZH\Omega = 0, \quad \Omega \in A_n(\mathbb{R}).$$

Reconstruction approach:

- If one can write $n^2 - 1$ independent linear constraints as above ("hyperplane condition") at a given point, the matrix $\tilde{A}S$ can be reconstructed from a generalization of the cross-product.
- Whenever $B = \tilde{A}S$ is known, $\tilde{\gamma} = \tilde{A}^2$ is reconstructed by setting

$$\tilde{\gamma} = BH^{-\frac{1}{2}}(BH^{-\frac{1}{2}})^T = BH^{-1}B^T, \quad H := \{H_{ij}\}_{1 \leq i, j \leq n}. $$
Stability

**Theorem (Uniqueness and stability for \( \tilde{\gamma} \))**

Over \( \Omega \subset X \) where the hyperplane condition is satisfied, \( \tilde{\gamma} \) is uniquely determined, with stability

\[
\| \tilde{\gamma} - \tilde{\gamma}' \|_{L^\infty(\Omega)} \leq C \| H - H' \|_{W^{1,\infty}(X)}.
\]


▷ **Explicit reconstruction.** Loss of one derivative on \( \tilde{\gamma} \).

Microlocal analysis on the linearized problem in [Bal-Guo-M., preprint] shows that the loss of one derivative is optimal.

Remarks on the validity conditions:
- Can be fulfilled locally for \( C^{1,\alpha} \) tensors (Runge approximation).
- Their robustness for the reconstruction algorithm is however highly sensitive to noise.
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Reconstruction of the scalar factor

**Problem:** Reconstruct $\beta := (\det \gamma)^{\frac{1}{n}}$ ($\tilde{\gamma}$ known) from the knowledge of $J_i(x) = \gamma \nabla u_i$, where $u_i$ solves

$$\nabla \cdot (\gamma \nabla u_i) = 0, \quad u_i|_{\partial \Omega} = g_i, \quad i = 1, 2.$$ 

**Assumption:** $(\nabla u_1, \nabla u_2)$ linearly independent.

$$\inf_{\Omega} \left( |J_1|^2 |J_2|^2 - (J_1 \cdot J_2)^2 \right) \geq c_1 > 0 \quad \text{over some } \Omega \subset X,$$

Rewrite the equations $d(\gamma^{-1} J_i) = d^2 u_i = 0$ ($i = 1, 2$) as

$$\nabla \log \beta \wedge (\tilde{\gamma}^{-1} J_i) = d(\tilde{\gamma}^{-1} J_i), \quad i = 1, 2,$$

and derive again a transport equation for $\log \beta$.

▷ Reconstruction approach and Lipschitz stability property similar to the power density case.
Reconstruction of $\tilde{\gamma}$ via a coupled elliptic system

Start from $(u_1, \ldots, u_n)$ a "basis" of solutions and let $u_{n+1}, u_{n+2}$ two additional solutions generating vector fields

$$Z_{i,j} = \nabla \frac{\det(\nabla u_1, \ldots, \nabla u_{n+i}, \ldots, \nabla u_n)}{\det(\nabla u_1, \ldots, \nabla u_n)}, \quad i = 1, 2, \quad 1 \leq j \leq n.$$

The previously derived equation $\sum_{j=1}^{n} Z_{i,j} \wedge \nabla u_j = 0$ can be written as

$$Z_{i,p}^* \cdot \nabla u_q = Z_{i,q}^* \cdot \nabla u_p, \quad i = 1, 2, \quad 1 \leq p < q \leq n.$$

With these equations and under some additional conditions, one can derive a system of the form

$$\nabla \cdot (A \nabla u_i) + W_{ij} \cdot \nabla u_j = 0, \quad (X), \quad u_i|_{\partial X} = g_i,$$

where $A$ is unif. elliptic and $(A, W_{ij})$ are constructed from the vector fields $Z_{i,j}$. 
Reconstruction procedure

The system above is **Fredholm**. If it is injective, one may reconstruct \((u_1, \ldots, u_n)\), then \(\gamma = [J_1| \ldots |J_n][\nabla u_1| \ldots |\nabla u_n]^{-1}\).

**Important remark:** Although derived in the context of current densities, this algorithm only uses the vector fields \(Z_i\) as data, which may be generated from a large class of measurements.

**Stability:** similar to the power densities problem (loss of one derivative). Improved stability for \(d\gamma^{-1}\) due to the identity

\[
\partial_q \gamma^{pl} - \partial_p \gamma^{ql} = J^{jl}(\gamma^{qj} \partial_p J_{ji} - \gamma^{pj} \partial_q J_{ji}), \quad 1 \leq l \leq n, \quad 1 \leq p < q \leq n.
\]

(derived from \(d(\gamma^{-1} J_i) = 0, \quad 1 \leq i \leq n\).
Conclusive remarks

- Conductivity tensors can be **fully reconstructed** from power densities or current densities.
- **Scalar factor:**
  - reconstructible via solving a transport equation,
  - **Lipschitz** stability: no loss of scales
- **Anisotropic structure:**
  - reconstructible via explicit pointwise algebraic formula (power densities), possibly after solving a coupled elliptic system (current densities).
  - the latter approach appears to be generalizable to a **large class of internal measurements**!
  - **Hölder** stability: loss of one derivative (mild ill-posedness).
  - Additional Lipschitz stability is obtained on some functionals of $\gamma$ ($d\gamma^{-1}$ in the case of current densities)
Thank you!

References and slides available at
http://www.math.washington.edu/~fmonard/research.html