Efficient tensor tomography in fan-beam coordinates

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Outline

1 Introduction

2 Equivalence of range characterizations

3 Tensor Tomography
The Radon transform

First considered and inverted by Johann Radon in 1917. \( \Omega \subset \mathbb{R}^2 \) bounded, \( f \) with support in \( \Omega \).

\[
f(x) \underset{R}{\rightarrow} \quad Rf(s, \theta) = \int_{\mathbb{R}} f(s\hat{\theta} + t\hat{\theta}^\perp) \, dt,
\]

\[
\hat{\theta} := \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad (s, \theta) \in \mathbb{R} \times S^1.
\]
Efficient tensor tomography in fan-beam coordinates

Introduction

Application to transmission tomography

X-ray Computerized Tomography

Applications: tumor detection, bone damage, . . .

Mathematical model:
Radon transform

⇔
Vector/tensor tomography

$F$ can be a vector field or a symmetric $m$-tensor

$$\text{IF}(x, v) = \int_0^{\tau(x,v)} F(\gamma_{x,v}(t), \otimes^m \dot{\gamma}_{x,v}(t)) \, dt, \quad (x, v) \in \partial_+ SM$$

Natural kernel (e.g. for $m = 1$): if $F = df$ with $f|_{\partial M} = 0$, then $\text{IF} = 0$. Similar kernels for any $m \geq 1$. [Sharafutdinov ’94]

Connections:

- $m = 1$: Doppler tomography [Anikonov–Romanov ’97] [Holman–Stefanov ’10]
- $m = 0, 2$: linearized boundary rigidity [Sharafutdinov ’07]
- $m = 4$: tomography in slightly anisotropic elastic media. [Sharafutdinov ’94]

Implementations: Schuster, Katsevich, Kazantzev, Bukhgeim, Derevtsov, Svetov, ...
1 Introduction

2 Equivalence of range characterizations

3 Tensor Tomography
Main theoretical difference: PG enjoys the Fourier Slice theorem, which allows for a proper, efficient regularization theory. Regularized inversions are approximate in fan-beam. [Natterer ’01]

Yet,

- working with fan-beam data is of interest (e.g. projection onto range)
- on surfaces without symmetries, parallel geometry does not exist!
Parallel v/s fan-beam geometry

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- on surfaces without symmetries, parallel geometry does not exist!
The classical moment conditions

Parallel geometry: $\mathcal{R} : \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}(\mathbb{R} \times S^1)$

$$\mathcal{R}f(s, \theta) = \int_{\mathbb{R}} f(-s\hat{\theta}^\perp + t\hat{\theta}) \, dt, \quad (s, \theta) \in \mathbb{R} \times S^1.$$ 

Moment conditions: Gelfand, Graev, Helgason, Ludwig

(i) $\mathcal{D}(s, \theta) = \mathcal{R}f(s, \theta)$ for some $f$ if

(ii) For $k \geq 0$, $p_k(\theta) := \int_{\mathbb{R}} s^k \mathcal{D}(s, \theta) \, ds = \sum_{k=-k}^{k} a_{k,k} e^{ik\theta}$.

(\text{ii}) $\int_{\mathbb{R}} \int_{S^1} \mathcal{D}(s, \theta)s^k e^{ik\theta} \, ds \, d\theta = 0$, $|p| > k$, $p - k$ even.
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$(ii')$ $\int_{S^1} \int_{\mathbb{R}} \mathcal{D}(s, \theta) s^k e^{ip\theta} \, ds \, d\theta = 0$, $|p| > k$, $p - k$ even.
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\( D(s, \theta) = \mathcal{R}f(s, \theta) \) for some \( f \) if

(i) \( D(s, \theta) = D(-s, \theta + \pi) \) for all \( (s, \theta) \in \mathbb{R} \times \mathbb{S}^1 \).

(ii) For \( k \geq 0 \),

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p_k(\theta) := \int_{\mathbb{R}} s^k D(s, \theta) \, ds = \sum_{\ell=-k}^{k} a_{\ell,k} e^{i\ell\theta}.
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The Pestov-Uhlmann range characterization

\((M, g)\) simple surface, inward boundary \(\partial_+ SM\). Define \(l_0 : C^\infty(M) \rightarrow C^\infty(\partial_+ SM)\) as

\[
l_0 f(x, v) = \int_0^{\tau(x, v)} f(\gamma_{x,v}(t)) \, dt,
\]

\((x, v) \in \partial_+ SM\).

Scattering relation \(s\). Define \(P := A_- H_- A_+\), where

- \(A_+ : C^\infty(\partial_+ SM) \rightarrow C^\infty(\partial SM)\) symmetrization w.r.t. \(s\).
- \(H_-\): odd Hilbert transform on the fibers of \(\partial SM\).
- \(A_-^* : C^\infty(\partial SM) \rightarrow C^\infty(\partial SM)\): \(A_-^* f(x, v) = f(x, v) - f(s(x, v))\).

Range characterization of \(l_0\):

\[
l_0(C^\infty(M)) = P_-(C_s^\infty(\partial_+ SM)).
\]

[Pestov-Uhlmann ’05]
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Efficient tensor tomography in fan-beam coordinates

Equivalence of range characterizations

Theorem (M., ’15)

The Pestov-Uhlmann range characterization is equivalent to the classical moment conditions in the Euclidean setting.

Sketch of proof: Fan-beam coordinates: $(\beta, \alpha) \in S^1 \times (-\frac{\pi}{2}, \frac{\pi}{2})$.

- Explicit scattering relation:
  
  $s(\beta, \alpha) = (\beta + \pi + 2\alpha, \pi - \alpha)$.

- Explicit construction of the SVD of $P_- : L^2_+ \rightarrow L^2_-$ via an ad hoc basis of symmetrized complex exponentials.

- Reparameterized moment conditions is equivalent to saying “$D \setminus Range \ P_-$”.
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\[ u'_{p,q} = e^{ip\beta}(e^{i(2q+1)\alpha} + (-1)^p e^{i(2(p-q)-1)\alpha}) \]
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Projecting noisy data onto range $I_0$ in fan-beam coordinates

Despite characterizing Range $I_0$, $P_-$ cannot be used to project noisy data onto the range of $I_0$. Define, instead, $C_- := \frac{1}{2}A^*H_-A_-$. The operator $Id + C_-^2$ is the $L^2$-orthogonal projection onto Range $P_-$ ($= \text{Range } I_0$).
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3. Tensor Tomography
The tensor tomography problem

For $f(x, \theta) \in C^\infty(M \times S^1)$, define

$$I_f(\beta, \alpha) = \int_0^{2 \cos \alpha} f(e^{i\beta}(1-te^{i\alpha}), \beta+\pi+\alpha) \, dt, \quad (\beta, \alpha) \in S^1 \times (-\pi/2, \pi/2).$$

Contains:

- $I_f = I_0 f$ whenever $f(x, \theta) = f(x)$.
- Tensor tomography over $m$-tensors:
  $$f(x, \theta) = \sum_{\ell=-m:2:m} f_\ell(x) e^{im\theta}.$$
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\( I : L^2(M \times S^1) \rightarrow L^2(\partial^+ SM) \) is continuous and **surjective**.
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Litterature

**Euclidean**
- Reconstruction of the solenoidal part in any dimension and tensor order [Sharafudtinov ’94]
- Characterization of solenoidal tensors, reconstruction methods: [Kazantzev-Bukhgeim ’06], [Derevtsov-Svetov ’15]
- A-analytic theory, general convex domains: [Sadiq-Sherzer-Tamasan ’14]

**Riemannian**
- TTP on simple surfaces: [Paternain-Salo-Uhlmann ’12]
- Range characterization: [Paternain-Salo-Uhlmann ’12]
- Non-trapping surfaces of revolution: [Sharafudtinov ’97]
- Higher dimensions, assuming a foliation condition [Stefanov-Uhlmann-Vasy ’14]
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The potential-solenoidal decomposition

\[ \mathbf{d} := \sigma \nabla \text{ “inner derivative”}. \] Any \( m + 1 \)-tensor decomposes uniquely into \( f = \mathbf{d}h + f^s \) with \( h|_{\partial M} = 0 \) and \( \delta f^s = 0 \).

[Sharafutdinov ’94] Since \( I(\mathbf{d}h) = 0 \), we have \( If = If^s \).

Natural question: how to reconstruct of \( f^s \) from \( If \)?

Past work:

- [Sharafutdinov ’94]: inversion formulas for \( f^s \), any order, any dimension, in Euclidean free space. Based on inverting \( \delta \mathbf{d} \), an operator whose expression depends on the tensor order.

- [Kazantzev-Bukhgeim ’04]: Full SVD description of \( I_m : L^2(S^m_{sol}) \rightarrow L^2(\partial_+ SM) \). \( m \)-dependent.

- [Derevtsov-Svetov ’15]: express \( f^s \) in terms of a solenoidal potential \( h \), reconstruct \( h \) then \( f^s \). Too many differentiations and integrations.
Observations

\[ L^2(M \times S^1) = \bigoplus_{k \in \mathbb{Z}} \Omega_k, \quad \Omega_k := \{ f(x)e^{ik\theta}, \ f \in L^2(M) \}. \]
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Tensor Tomography

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\[ I(f_{-2}(x)e^{-2i\theta} + f_0(x) + f_2(x)e^{2i\theta}) = I(2\text{-tensors}) \]
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\[ I(f_{-2}(x)e^{-2i\theta} + f_0(x) + f_2(x)e^{2i\theta}), \quad \overline{\partial}_zf_2 = \partial_zf_{-2} = 0 \]
A different decomposition: example on two-tensors

Example for $m$ even (similar result for $m$ odd).

**Theorem (M., ’15)**

Let $m = 2p$. For any $f \in L^2(M \times S^1) \cap S^m$, there exists a unique $g \in L^2(M \times S^1) \cap S^m$ such that

1. $If = Ig$,
2. $g = \sum_{k=-m:2:m} g_k(x) e^{ik\theta}$, $\overline{\partial_z g_{2k}} = \partial_z g_{-2k} = 0$, $1 \leq k \leq p$.
3. $\|g\|_{L^2(SM)} \leq C_m \|f\|_{L^2(SM)}$ for some constant $C_m$ indep. of $f, g$.

Analogue exists on simple surfaces.

**Theorem (M., ’15)**

The ray transforms of the components of $g$ live on orthogonal subspaces of $L^2(\partial_{+} SM)$. 

\text{Efficient tensor tomography in fan-beam coordinates}
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Tensor Tomography
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Tensor Tomography

Decomposition in data space

Even tensors

\[ l_0 f = lf \]

Odd tensors

\[ l_\perp h = l(*dh) \]
Reconstructions

Case of even tensors. The odd case is similar.

**Theorem (M., ’15)**

Let $m = 2p$. Denote $\mathcal{D} = I f = I g$ with $f, g$ as above. The functions $g_{2k}$ can be reconstructed as follows:

$$
g_0 = \frac{1}{8\pi} I_{\|}^\# A^* \mathcal{H} A_- \mathcal{D}, \quad \text{and for} \quad 1 \leq k \leq n,$$

$$
g_{2k}(z) = \frac{1}{2\pi^2} \int_{S^1} \frac{e^{-i2k\beta}}{(1 - ze^{-i\beta})^2} \int_{-\pi/2}^{\pi/2} \mathcal{D}(\beta, \alpha) e^{i(1-2k)\alpha} \ d\alpha \ d\beta,$$

$$
g_{-2k}(z) = \frac{1}{2\pi^2} \int_{S^1} \frac{e^{i2k\beta}}{(1 - \bar{z} e^{i\beta})^2} \int_{-\pi/2}^{\pi/2} \mathcal{D}(\beta, \alpha) e^{i(-1+2k)\alpha} \ d\alpha \ d\beta.$$

Very efficient to compute. Does not differentiate. **Order-blind**
Example of a 2-tensor $f = f_0 + 2f^r_2 \cos(2\theta) - 2f^i_2 \sin(2\theta)$
Numerical examples

Reconstructed equivalent:

\[ |lf - lg| \]
Numerical examples

vector field: $f = f_1^r \cos(\theta) - f_1^i \sin(\theta) + f_3^r \cos(3\theta) - f_3^i \sin(3\theta)$
3-tensor: $g = \star dg_0 + g^r_3 \cos(3\theta) - g^i_3 \sin(3\theta)$

$g_0$ is one degree smoother than $f$. 
Summary:

- equivalence of range characterizations,
- fast projection onto the range of $l_0$ in fan-beam coordinates.
- an *ad hoc* decomposition for tensor tomography reconstruction.
- a toy model toward understanding fan-beam coordinates and tensor tomography on surfaces.

Thank you

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