Efficient tensor tomography in fan-beam coordinates

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Outline

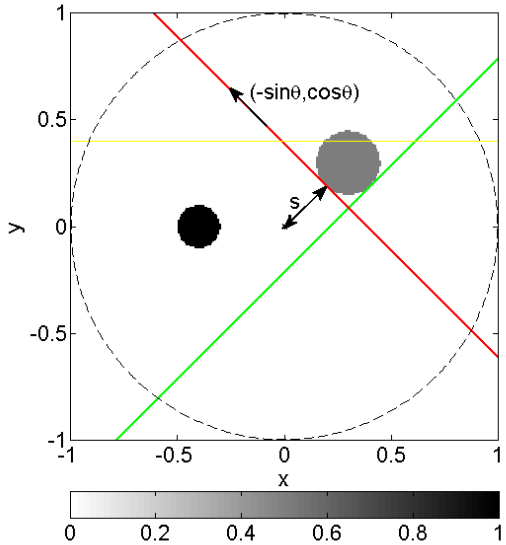
The Radon transform

First considered and inverted by Johann Radon in 1917.

$\Omega \subset \mathbb{R}^2$ bounded, f with support in Ω.

$$f(x) \quad \overset{R}{\mapsto} \quad Rf(s,\theta) = \int_{\mathbb{R}} f(s\hat{\theta} + t\hat{\theta}^{\perp})\, dt,$$

$$\hat{\theta} := \left(\begin{smallmatrix} \cos\theta \\ \sin\theta \end{smallmatrix}\right), \quad (s,\theta) \in \mathbb{R} \times \mathbb{S}^1.$$



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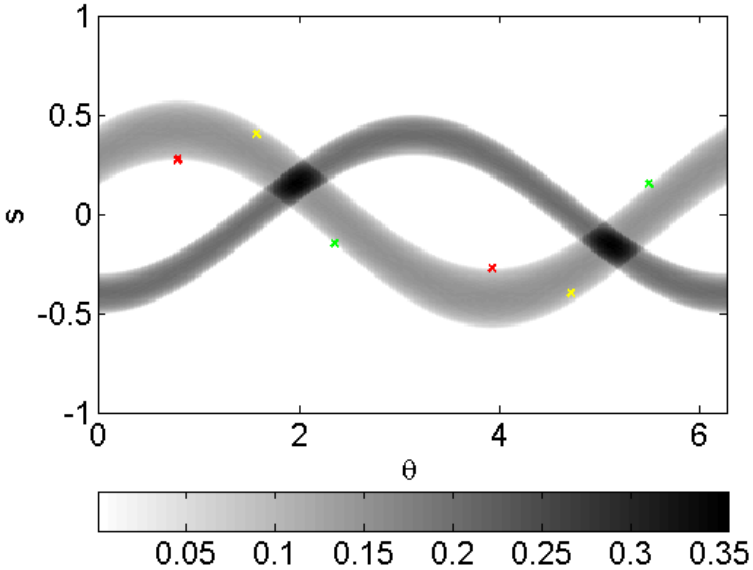
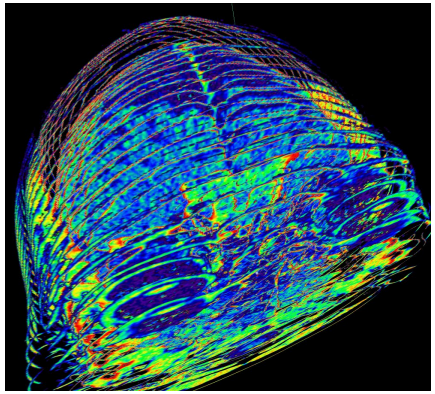


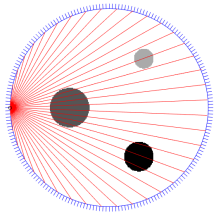
Application to transmission tomography

X-ray Computerized Tomography



Applications: tumor detection, bone damage, . . .

Mathematical model:
Radon transform



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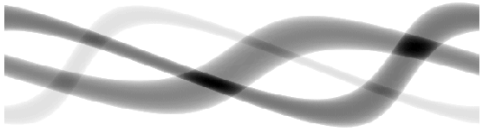


Vector/tensor tomography

F can be a vector field or a symmetric m-tensor

$$IF(\mathrm{x}, v) = \int_0^{\tau(\mathrm{x},v)} F(\gamma_{\mathrm{x},v}(t), \otimes^m \dot{\gamma}_{\mathrm{x},v}(t))\, dt, \quad (\mathrm{x}, v) \in \partial_+ SM \quad .$$

Natural kernel (e.g. for $m = 1$): if $F = df$ with $f|_{\partial M} = 0$, then $IF = 0$. Similar kernels for any $m \geq 1$. [Sharafutdinov '94]

Connections:

- $m = 1$: Doppler tomography [Anikonov-Romanov '97] [Holman-Stefanov '10]
- $m = 0, 2$: linearized boundary rigidity [Sharafutdinov '07]
- $m = 4$: tomography in slightly anisotropic elastic media. [Sharafutdinov '94]
- Implementations: Schuster, Katsevich, Kazantzev, Bukhgeim, Derevtsov, Svetov, ...

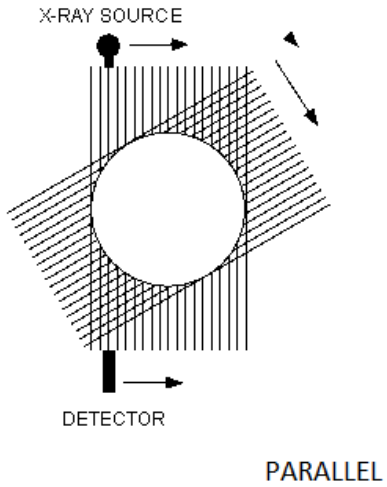
Outline

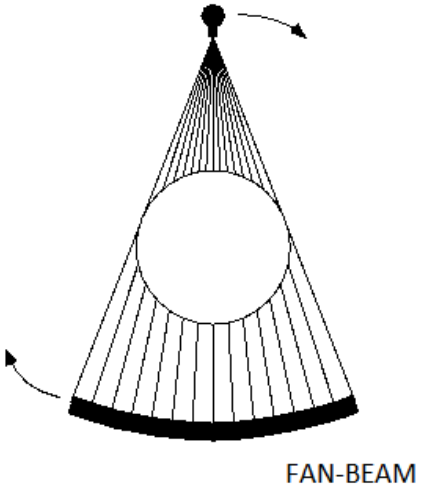
Parallel v/s fan-beam geometry

Main theoretical difference: PG enjoys the Fourier Slice theorem, which allows for a proper, efficient regularization theory. Regularized inversions are approximate in fan-beam. [Natterer '01]

Yet,

- working with fan-beam data is of interest (e.g. projection onto range)
- on surfaces without symmetries, parallel geometry does not exist !





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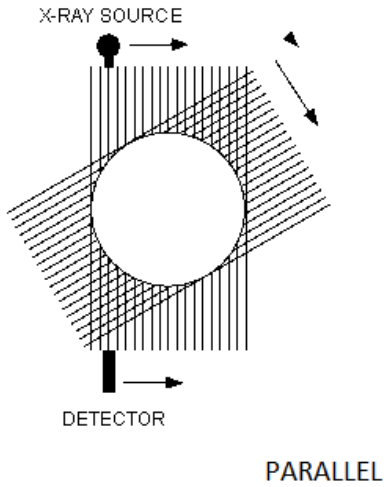

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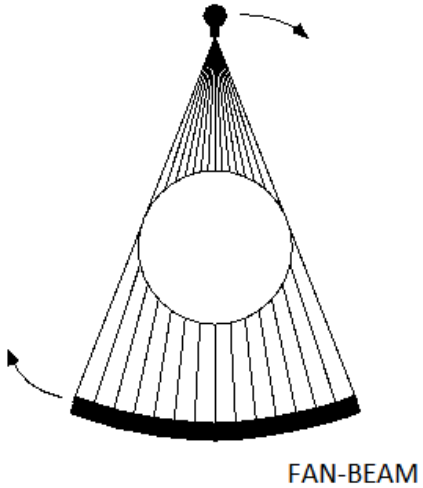
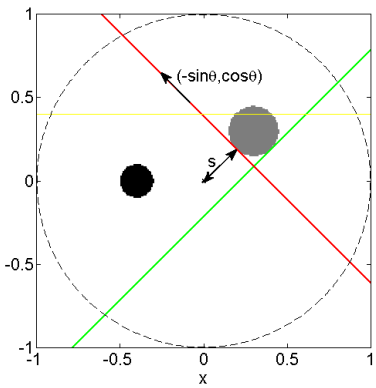


The classical moment conditions

Parallel geometry: $\mathcal{R} : \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}(\mathbb{R} \times \mathbb{S}^1)$

$$\mathcal{R}f(s,\theta) = \int_{\mathbb{R}} f(-s\hat{\theta}^{\perp} + t\hat{\theta})\, dt, \quad (s,\theta) \in \mathbb{R} \times \mathbb{S}^1.$$



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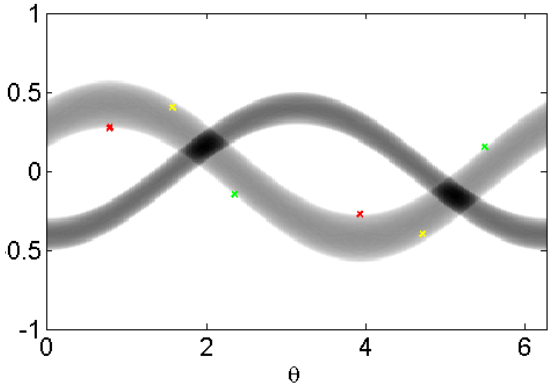

Moment conditions: Gelfand, Graev, Helgason, Ludwig

$\mathcal{D}(s,\theta) = \mathcal{R}f(s,\theta)$ for some f if

(i) $\mathcal{D}(s,\theta) = \mathcal{D}(-s,\theta+\pi)$ for all $(s,\theta) \in \mathbb{R} \times \mathbb{S}^1$,

(ii) For $k \geq 0$, $p_k(\theta) := \int_{\mathbb{R}} s^k \mathcal{D}(s,\theta)\, ds = \sum_{\ell=-k}^{k} a_{\ell,k} e^{ik\theta}$,

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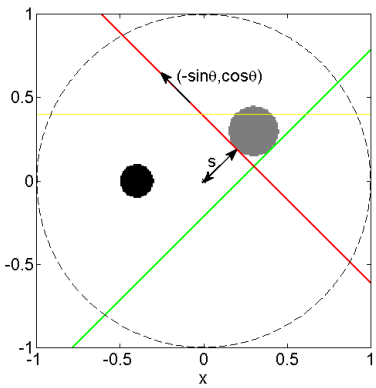

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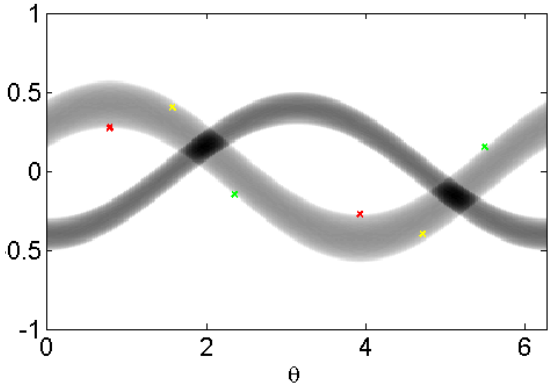

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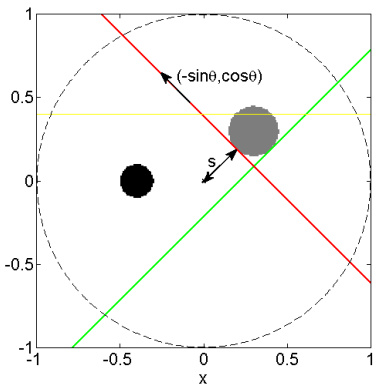

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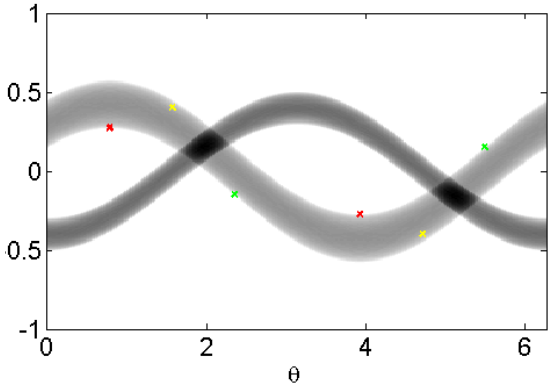

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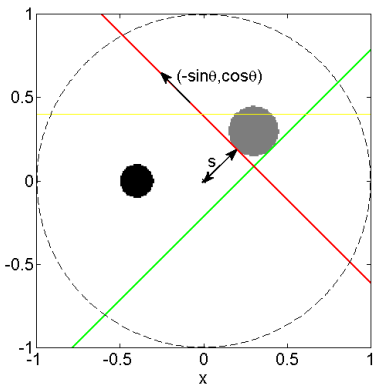

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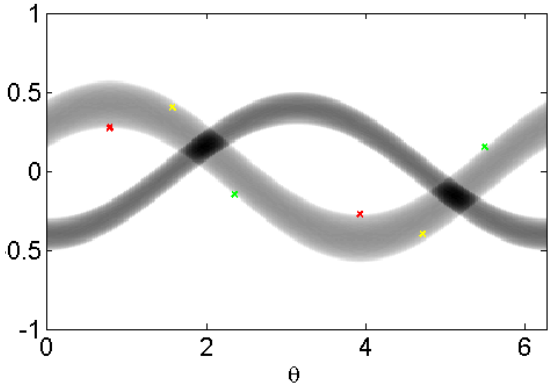

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The Pestov-Uhlmann range characterization

(M, g) simple surface, inward boundary $\partial_+ SM$. Define $I_0 : C^\infty(M) \to C^\infty(\partial_+ SM)$ as

$$I_0 f(x, v) = \int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\, dt, \qquad (x, v) \in \partial_+ SM.$$

Scattering relation s. Define $P := A_-^* H_- A_+$, where

- $A_+ : C^\infty(\partial_+ SM) \to C^\infty(\partial SM)$ symmetrization w.r.t. s.
- H_-: odd Hilbert transform on the fibers of ∂SM.
- $A_-^* : C^\infty(\partial SM) \to C^\infty(\partial SM)$: $A_-^* f(x, v) = f(x, v) - f(s(x, v))$.

Range characterization of I_0:

$$\boxed{I_0(C^\infty(M)) = P_-(C_s^\infty(\partial_+ SM)).}$$

[Pestov-Uhlmann '05]

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Equivalence of ranges characterizations

Theorem (M., '15)

The Pestov-Uhlmann range characterization is equivalent to the classical moment conditions in the Euclidean setting.

Sketch of proof: Fan-beam coordinates: $(\beta, \alpha) \in \mathbb{S}^1 \times (\frac{-\pi}{2}, \frac{\pi}{2})$.

- Explicit scattering relation: $s(\beta, \alpha) = (\beta + \pi + 2\alpha, \pi - \alpha)$.
- Explicit construction of the SVD of $P_- : L^2_- \to L^2_+$ via an ad hoc basis of symmetrized complex exponentials.
- Tensor (anti-)symmetrized moments of the range equivalent to moments of the range.

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- Reparameterized moment conditions is equivalent to saying "$\mathcal{D} \perp$ Range P_-".

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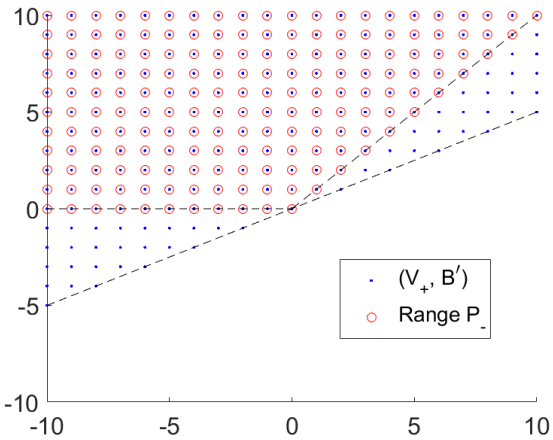

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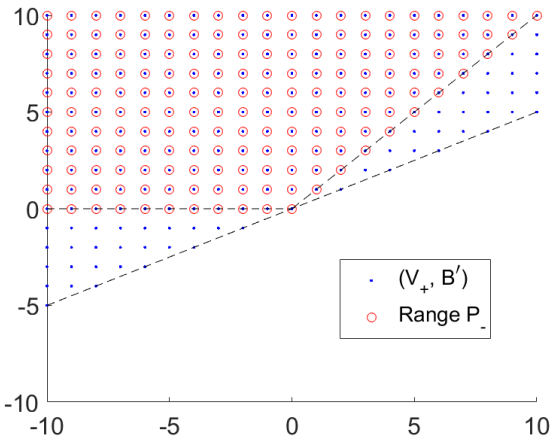

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Projecting noisy data onto range I_0 in fan-beam coordinates

Despite characterizing Range I_0, P_- cannot be used to project noisy data onto the range of I_0. Define, instead, $C_- := \frac{1}{2} A_-^* H_- A_-$.

The operator $Id + C_-^2$ is the L^2-orthogonal projection onto Range P_- (= Range I_0).

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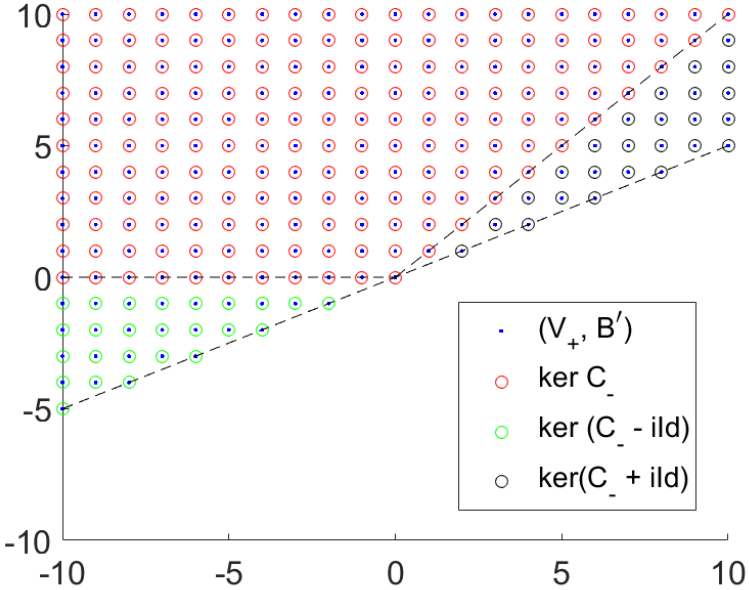

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Contains:

- $If = I_0 f$ whenever $f(x,\theta) = f(x)$.
- Tensor tomography over m-tensors: $f(x,\theta) = \sum_{\ell=-m:2:m} f_\ell(x) e^{im\theta}$.

$I : L^2(M \times \mathbb{S}^1) \to L^2(\partial_+ SM)$ is continuous and **surjective**.

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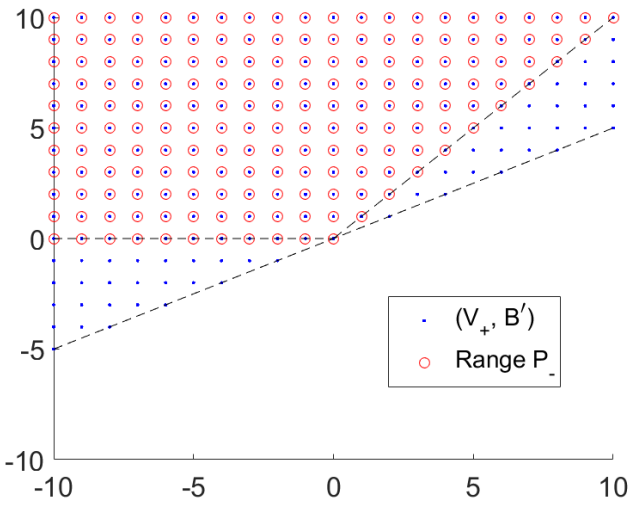

Litterature

Euclidean

- Reconstruction of the solenoidal part in any dimension and tensor order [Sharafudtinov '94]
- Characterization of solenoidal tensors, reconstruction methods: [Kazantzev-Bukhgeim '06], [Derevtsov-Svetov '15]
- A-analytic theory, general convex domains: [Sadiq-Sherzer-Tamasan '14]

Riemannian

- TTP on simple surfaces: [Paternain-Salo-Uhlmann '12]
- Range characterization: [Paternain-Salo-Uhlmann '12]
- Non-trapping surfaces of revolution: [Sharafutdinov '97]
- Higher dimensions, assuming a foliation condition [Stefanov-Uhlmann-Vasy '14]

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The potential-solenoidal decomposition

$\mathbf{d} := \sigma\nabla$: "inner derivative". Any $m+1$-tensor decomposes uniquely into $f = \mathbf{d}h + f^s$ with $h|_{\partial M} = 0$ and $\delta f^s = 0$. [Sharafutdinov '94] Since $I(\mathbf{d}h) = 0$, we have $If = If^s$.

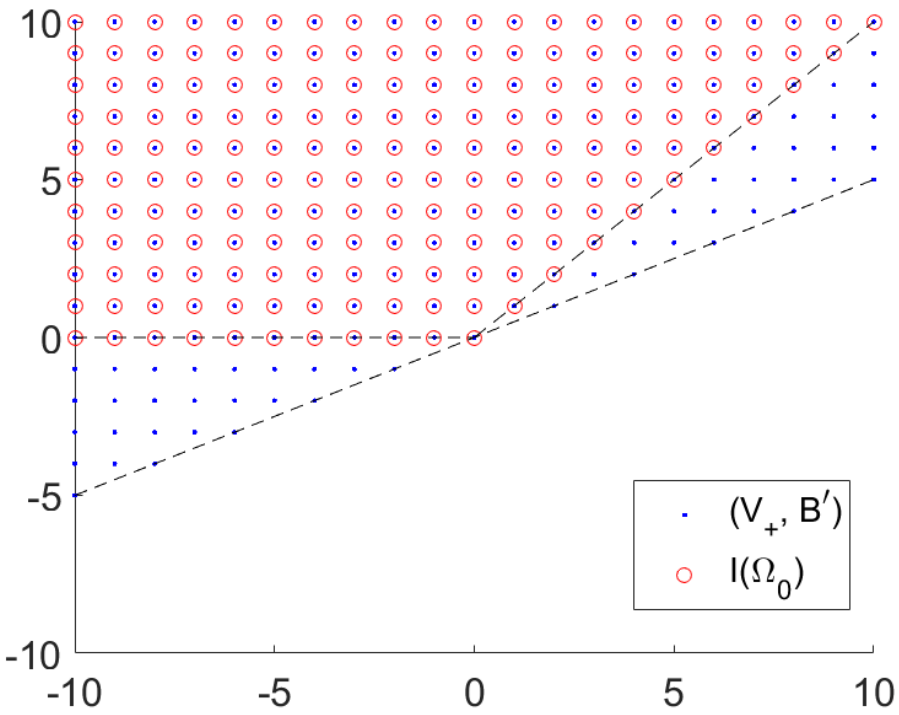
Natural question: how to reconstruct of f^s from If ?

Past work:

- [Sharafutdinov '94]: inversion formulas for f^s, any order, any dimension, in Euclidean free space. Based on inverting $\delta\mathbf{d}$, an operator whose expression depends on the tensor order.
- [Kazantzev-Bukhgeim '04]: Full SVD description of $I_m : L^2(S^m_{sol}) \to L^2(\partial_+ SM)$. m-dependent.
- [Derevtsov-Svetov '15]: express f^s in terms of a solenoidal potential h, reconstruct h then f^s. Too many differentiations and integrations.

Observations

$$L^2(M \times \mathbb{S}^1) = \bigoplus_{k \in \mathbb{Z}} \Omega_k, \qquad \Omega_k := \{f(x)e^{ik\theta}, \quad f \in L^2(M)\}.$$



$$I(f_0(x))$$

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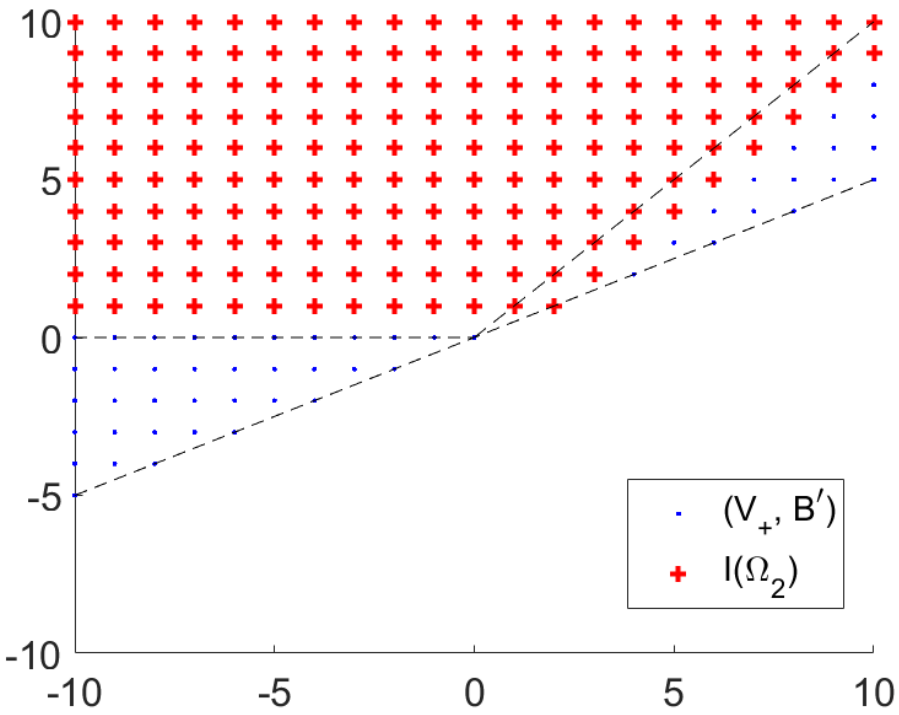

$$I(f_2(x)e^{2i\theta})$$

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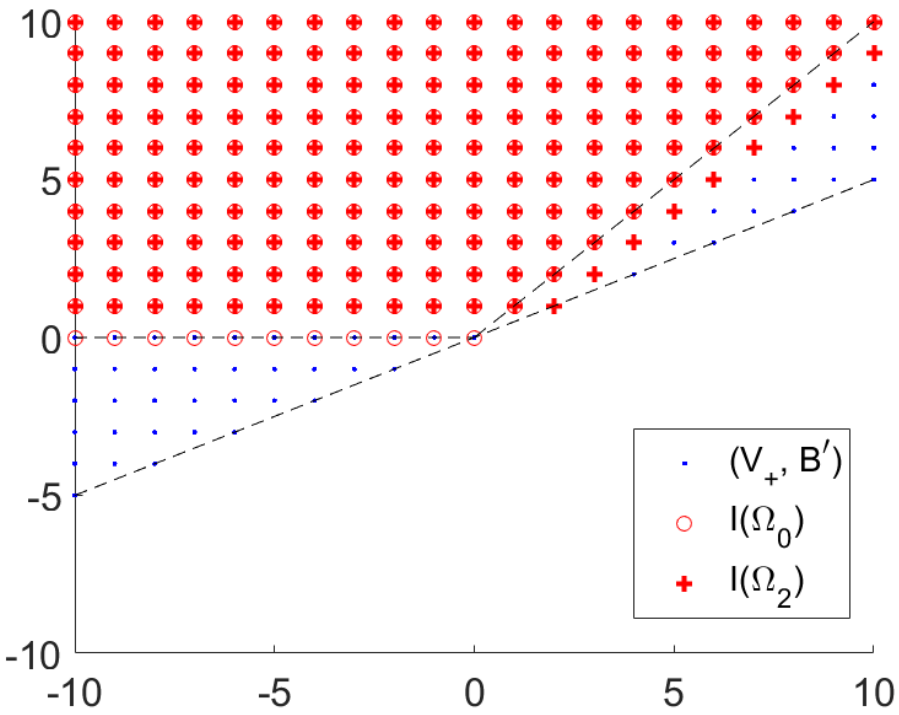

$$I(f_0(x) + f_2(x)e^{2i\theta})$$

Observations

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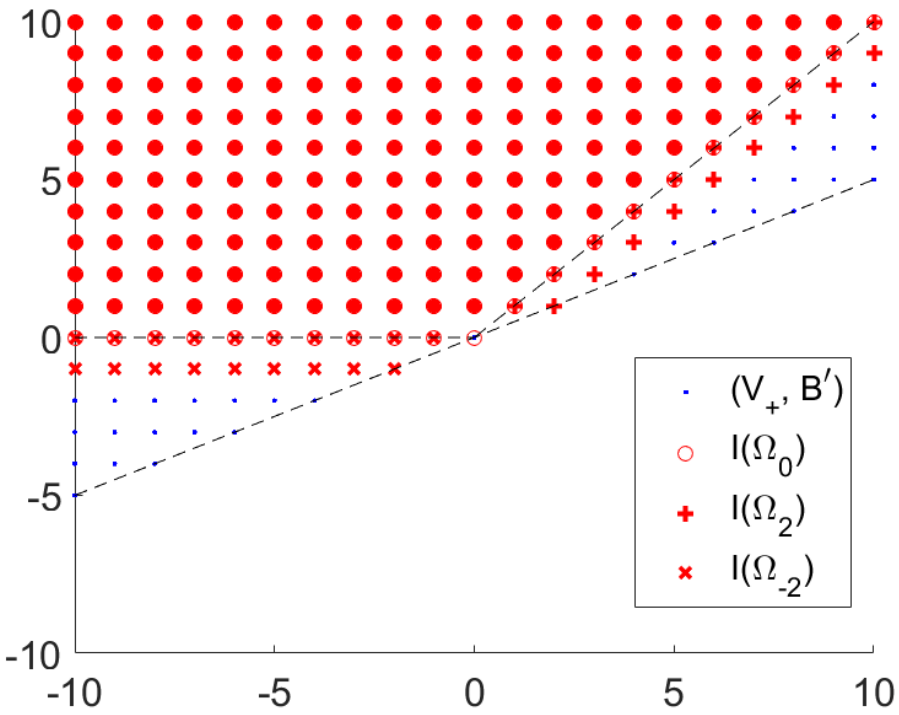

$$I(f_{-2}(x)e^{-2i\theta} + f_0(x) + f_2(x)e^{2i\theta}) = I(\text{2-tensors})$$

Observations

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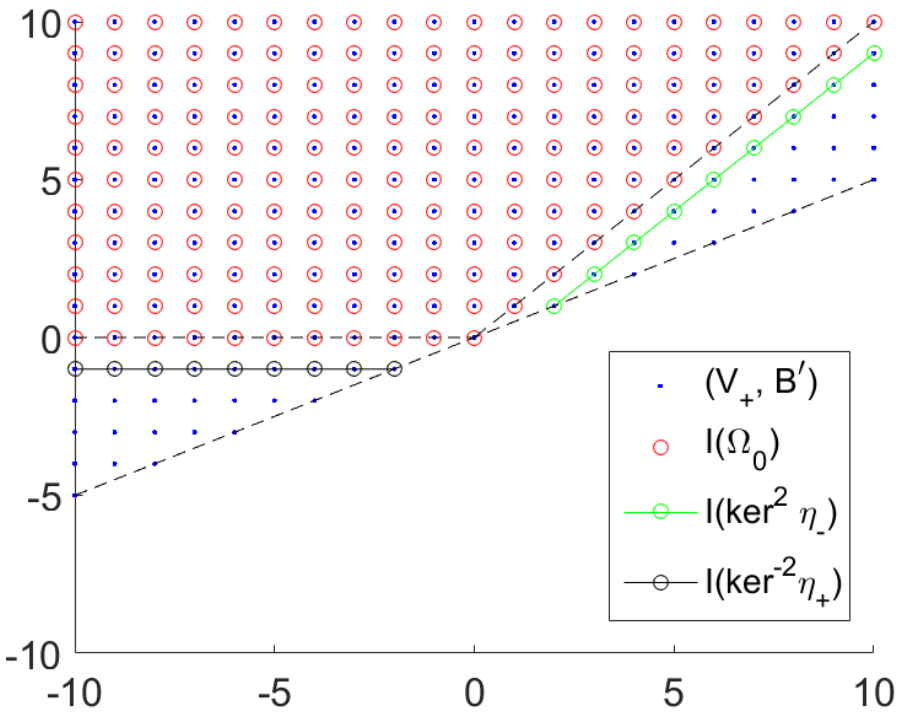

$$I(f_{-2}(x)e^{-2i\theta} + f_0(x) + f_2(x)e^{2i\theta}), \qquad \overline{\partial}_z f_2 = \partial_z f_{-2} = 0$$

A different decomposition: example on two-tensors

Example for m even (similar result for m odd).

Theorem (M., '15)

Let $m = 2p$. For any $f \in L^2(M \times \mathbb{S}^1) \cap S^m$, there exists a **unique** *$g \in L^2(M \times \mathbb{S}^1) \cap S^m$ such that*

1. $If = Ig$,
2. $g = \sum_{k=-m:2:m} g_k(x) e^{ik\theta}$, $\overline{\partial}_z g_{2k} = \partial_z g_{-2k} = 0$, $1 \leq k \leq p$.
3. $\|g\|_{L^2(SM)} \leq C_m \|f\|_{L^2(SM)}$ *for some constant C_m indep. of f, g.*

Analogue exists on simple surfaces.

Theorem (M., '15)

The ray transforms of the components of g live on orthogonal subspaces of $L^2(\partial_+ SM)$.

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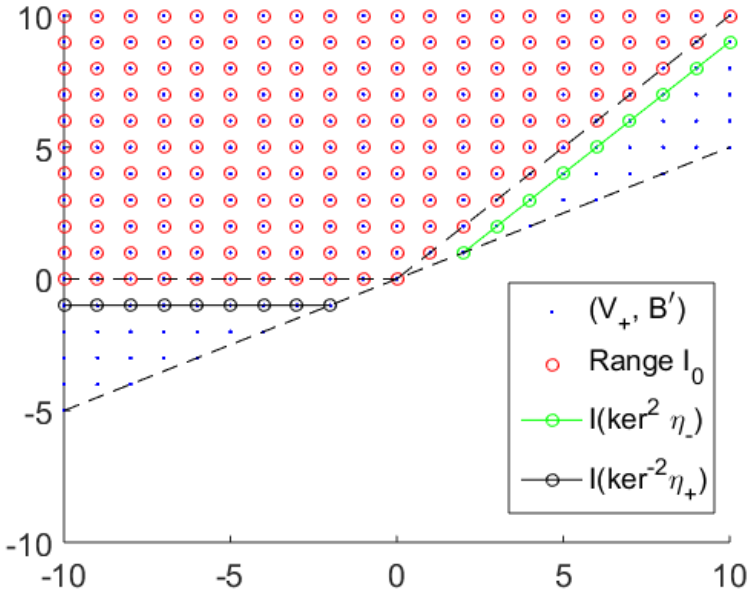
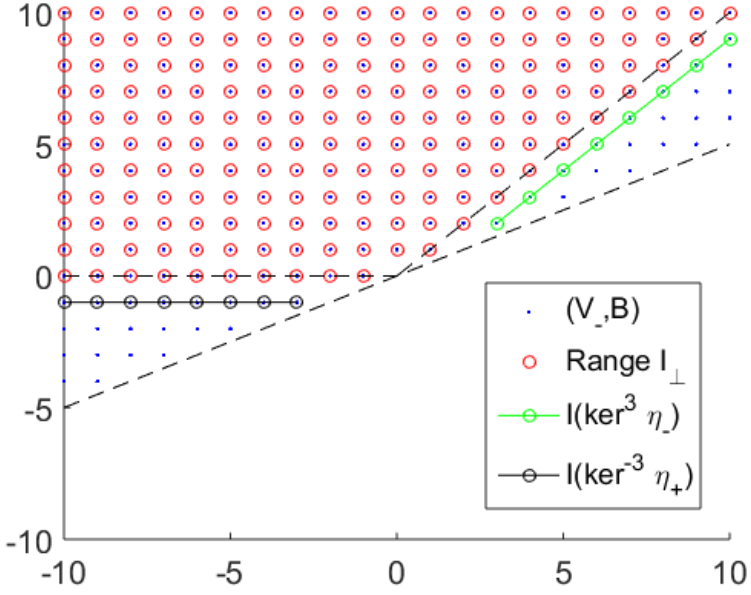
Decomposition in data space

Even tensors



$$I_0 f = If$$

Odd tensors



$$I_\perp h = I(*dh)$$

Reconstructions

Case of even tensors. The odd case is similar.

Theorem (M., '15)

Let $m = 2p$. Denote $\mathcal{D} = If = Ig$ with f, g as above. The functions g_{2k} can be reconstructed as follows:

$$g_0 = \frac{1}{8\pi} I_\perp^\sharp A_+^\star H A_- \mathcal{D}, \qquad \text{and for} \quad 1 \le k \le n,$$

$$g_{2k}(z) = \frac{1}{2\pi^2} \int_{\mathbb{S}^1} \frac{e^{-i2k\beta}}{(1 - ze^{-i\beta})^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathcal{D}(\beta, \alpha) e^{i(1-2k)\alpha} \, d\alpha \, d\beta,$$

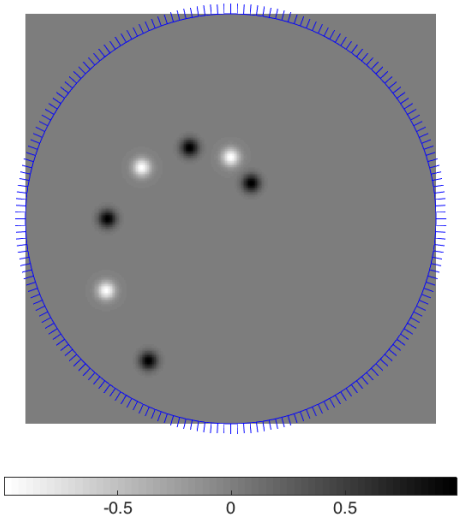
$$g_{-2k}(z) = \frac{1}{2\pi^2} \int_{\mathbb{S}^1} \frac{e^{i2k\beta}}{(1 - \bar{z}e^{i\beta})^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathcal{D}(\beta, \alpha) e^{i(-1+2k)\alpha} \, d\alpha \, d\beta.$$

Very efficient to compute. Does not differentiate. **Order-blind**

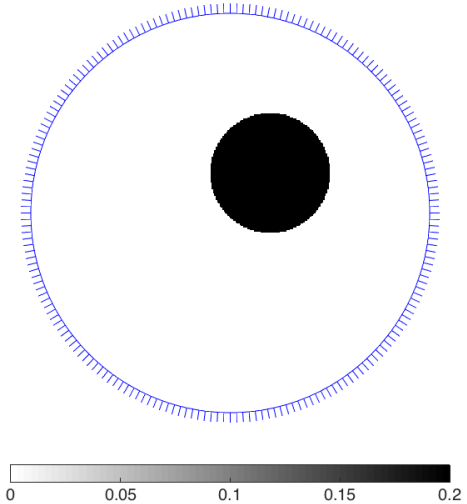
Numerical examples

Example of a 2-tensor $f = f_0 + 2f_2^r \cos(2\theta) - 2f_2^i \sin(2\theta)$

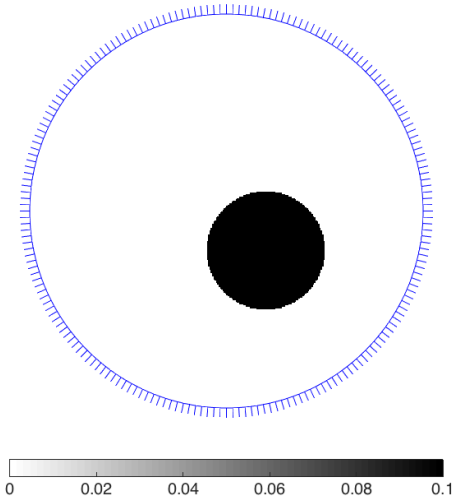
f_0



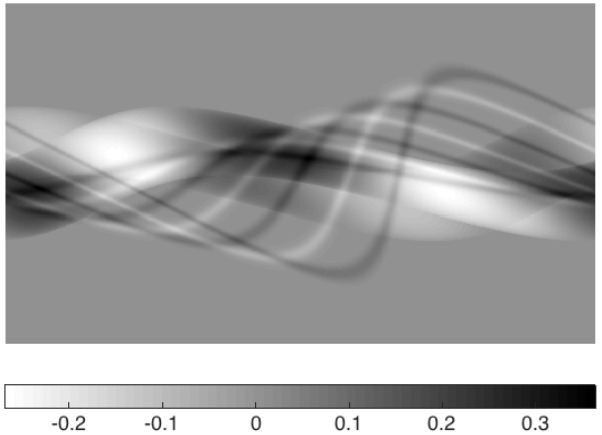
f_2^r



f_2^i



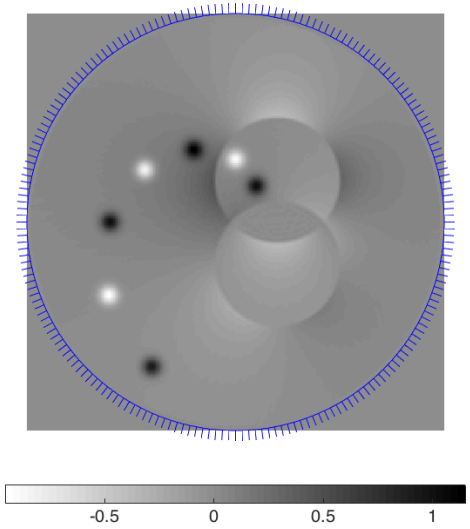
If



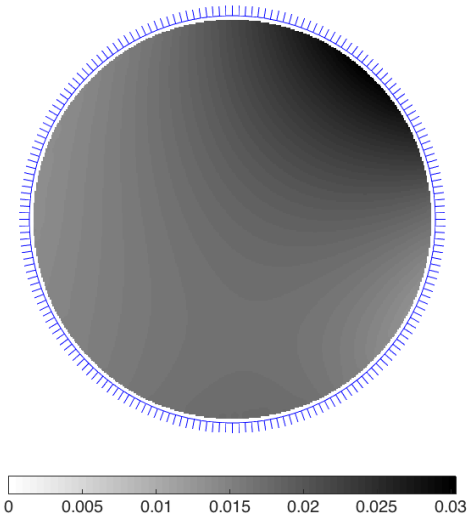
Numerical examples

Reconstructed equivalent:

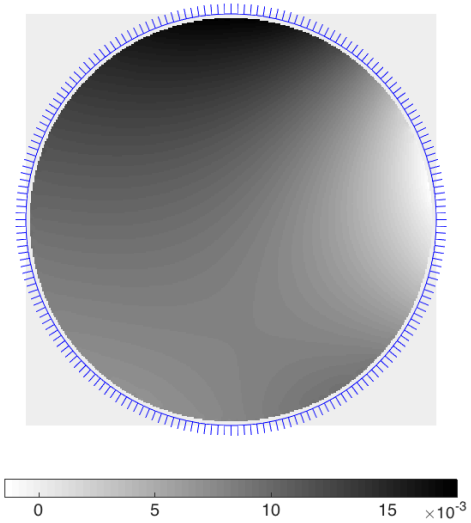
g_0



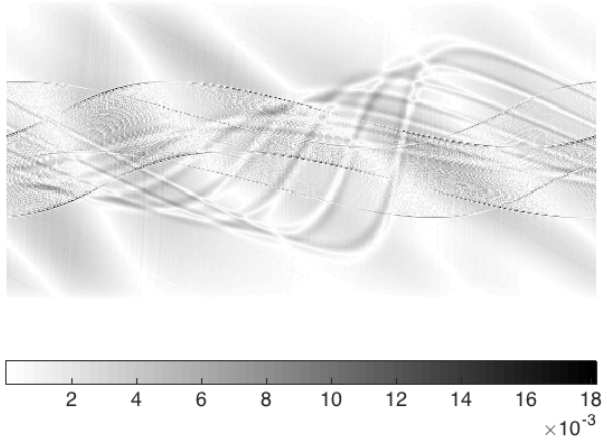
g_2^r



g_2^i



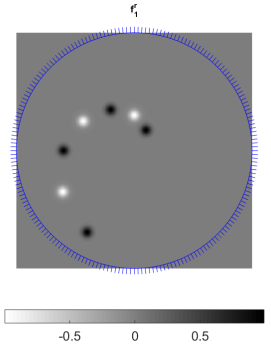
$|If - Ig|$



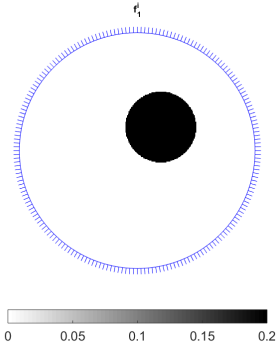
Numerical examples

vector field: $f = f_1^r \cos(\theta) - f_1^i \sin(\theta) + f_3^r \cos(3\theta) - f_3^i \sin(3\theta)$

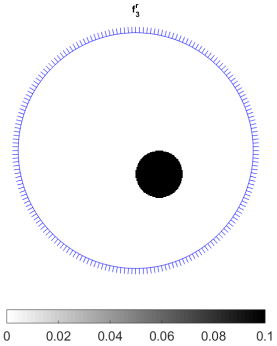
f_1^r



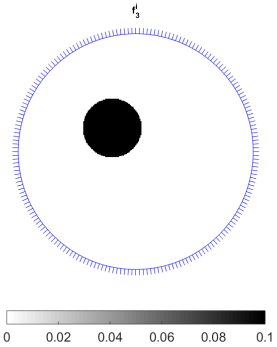
f_1^i



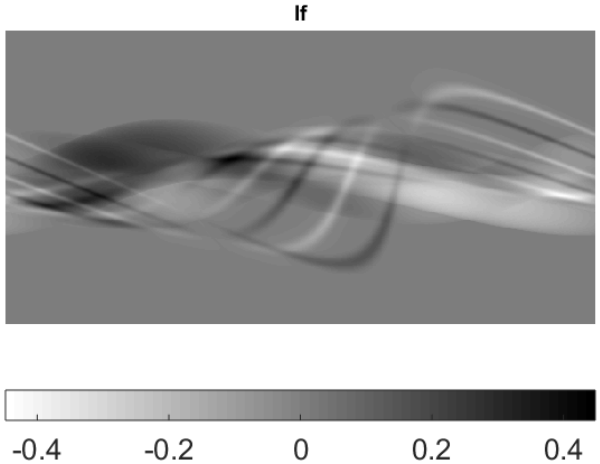
f_3^r



f_3^i



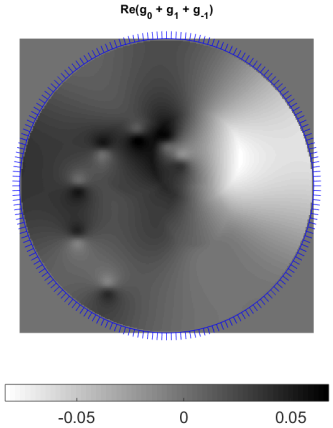
If



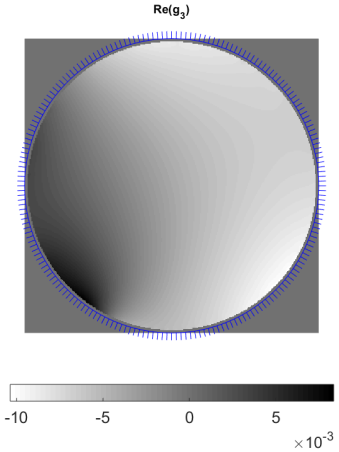
Numerical examples

3-tensor: $g = \star d g_0 + g_3^r \cos(3\theta) - g_3^i \sin(3\theta)$

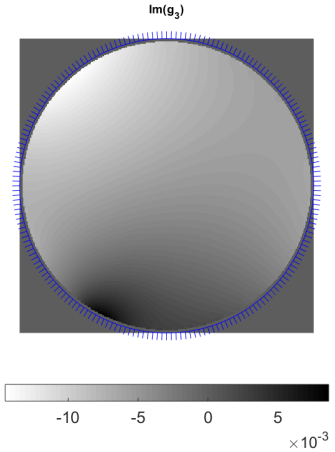
g_0



g_3^r



g_3^i



g_0 is one degree smoother than f.

Summary:

- equivalence of range characterizations,
- fast projection onto the range of I_0 in fan-beam coordinates.
- an *ad hoc* decomposition for tensor tomography reconstruction.
- a toy model toward understanding fan-beam coordinates and tensor tomography on surfaces.

Thank you

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• *Efficient tensor tomography in fan-beam coordinates*, Inv. Probl. Imaging **10**:2 (2016).

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