Toward imaging modalities with high(er) resolution

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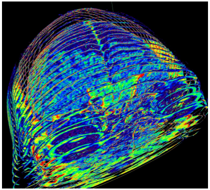
Topic du jour

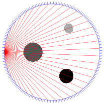
Inverse problems and non-invasive imaging

X-ray Computerized Tomography

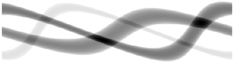


Applications: tumor detection, bone damage, ...

Mathematical model:
Radon transform



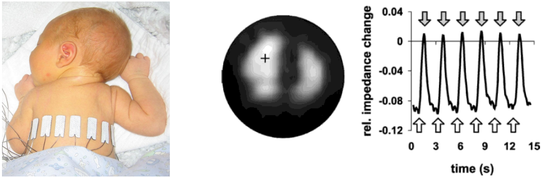
$\Leftrightarrow$



Topic du jour

Inverse problems and non-invasive imaging

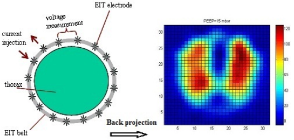
EIT (conductivity imaging)



[Heinrich et al '06]

Applications: lung monitoring, tumor detection, geophysical prospection. . .

Mathematical model: Calderón's problem



Analysis of inverse problems

Formally: Find $x \in \mathfrak{X}$ such that $y = \mathfrak{M}(x)$,
given **data** $y \in \mathfrak{Y}$ and **model** $\mathfrak{M}$.

Questions to answer:

1. **Uniqueness** ? $\mathfrak{M}(x_1) = \mathfrak{M}(x_2) \overset{?}{\Longrightarrow} x_1 = x_2$
2. **Stability** ? (rules resolution in practice)
   Hilbert scale of stability:
   - well-posed problems ($\delta\mathfrak{M}$ small $\Longrightarrow$ δx small)
   - mildly ill-posed problems ($\delta\mathfrak{M}$ small $\Longrightarrow$ δx not too large)
   - severely ill-posed ($\delta\mathfrak{M}$ small $\Longrightarrow$ δx too large)
3. How to **invert** ? How to deal with **noise in data** ?
   - Explicit inversion, range characterization,
   - ART, minimization/regularization methods, statistical inversions. . .

Stability: two (linear) examples

$\mathfrak{M}_1[f](x) = \int_0^x f(t)\ dt,$ $\mathfrak{M}_2[f](x) = (\frac{1}{2\sqrt{\pi T}} e^{-\frac{x^2}{4T}} * f)(x).$

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Both are injective. . .

$f(x) = \frac{d}{dx}\mathfrak{M}_1[f](x),$ $f(x) = \mathcal{F}^{-1}_{\xi\to x}(e^{T\xi^2}\mathcal{F}_{x\to\xi}\mathfrak{M}_2[f]).$

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...but with very different stability !

$\|f-g\|_{L^2(\mathbb{R})} \leq \|\mathfrak{M}_1[f] - \mathfrak{M}_1[f]\|_{H^1(\mathbb{R})}.$

▶ **Mildly** ill-posed.

No (s, p, C) such that

$\|f-g\|_{H^s(\mathbb{R})} \leq C\|\mathfrak{M}_2[f] - \mathfrak{M}_2[g]\|_{H^{s+p}(\mathbb{R})}.$

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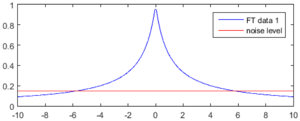
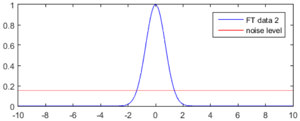
- **Severely** ill-posed.

Resolution on f from $\mathfrak{M}_2$ decreases much faster with noise level.





Medical imaging modalities

High stability, low contrast:

Ultrasound

MRI

CT

Hyperbolic PDEs: propagate singularities well.

Medical imaging modalities

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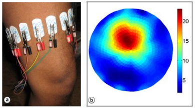
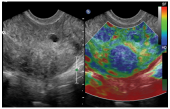
Low stability, high contrast:

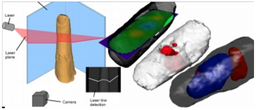
EIT



U.S. Elastography



Optical Diffuse Tomography.



Elliptic PDEs: forward model highly regularizing.

Motivation

Despite the poor resolution...

- **mechanical**, **optical** and **electrical** properties of tissues display **good contrast** for e.g. tumor detection, lung activity monitoring.
- Cheaper, more portable, less harmful...

Problem

How to **improve the stability** *of severely ill-posed problems ?*

Two ideas:

1. **Increase the dimensionality of the measurements.** Allows for more informative measurements.
2. **Couple** two physical models to achieve high contrast and high resolution: **hybrid methods**.

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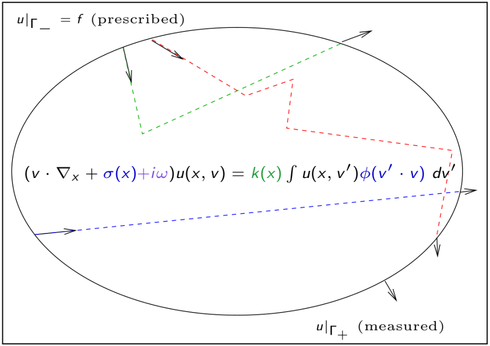
Outline

The transport equation (or radiative transfer)



Two regimes: Stationary or time-harmonic.

Applications:

- Optical Tomography (NIR photon transport).
- SPECT

Forward theory: [Dautray-Lions '93, Mokhtar-Kharroubi '97, Stefanov-Uhlmann '08]

$$u_f(x,v) = \mathcal{J}f + \mathcal{K}\mathcal{J}f + \sum_{m=2}^{\infty} \mathcal{K}^m \mathcal{J}f. \qquad \text{(scattering series)}$$

Inverse problems based on the transport regime

Angularly averaged measurements:

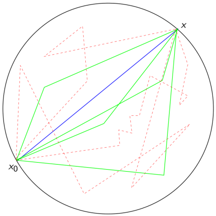
$$\mathcal{M}(f,x) = \int_{v\cdot\nu>0} u_f|_{\Gamma_+}(x,v)|v\cdot\nu|dS(v) = \mathcal{M}_0(f,x) + \mathcal{M}_1 k(f,x) + \mathcal{M}_{2+}[k](f,x)$$

Problem

With (σ,ϕ) known and $f = f(x)$, does $\mathcal{M}(f,x)$ determine $k(x)$ uniquely ? stably ?

Inversion strategy: $f = \delta_{x_0}$, $(x_0, x) \in \partial X^2$

- Ballistic part: known from σ.
- In the **transport** regime:
  - Single scattering $\to k$
  - Multiple scattering $\equiv$ small.



Legend: Ballistic part, single scattering and multiple scattering.

Stationary case [Bal-Langmore-M., IPI '08]

Problem

Reconstruct $k(x)$ from $\mathcal{M}(f,g) = \int_{\partial X} \mathcal{M}(f,x) g(x)\, dx.$

$$\mathcal{M}_1 k(f,g) = \int_X k(x_1) Af(x_1) Ag(x_1)\, dx_1, \quad A = DLP + \mathcal{O}(\|\sigma\|_\infty \|f\|_\infty \|g\|_\infty)$$

Idea: find (f,g) such that $\mathcal{M}_1 k(f,g) \approx \hat{k}(\xi)$ and invert via Fourier transform.

Using f and g traces of (harmonic) **CGO solutions**,

i.e. $\exp((\xi + i\xi^\perp)\cdot x), \quad \xi \in \mathbb{R}^2,$ yields

$$\mathcal{M}(f_\xi, g_\xi) - \mathcal{M}_0(f_\xi, g_\xi) = \hat{k}(\xi) + \epsilon(\sigma, k, \xi),$$

$$|\epsilon(\sigma,k,\xi) - \epsilon(\sigma,k',\xi)| \le C\|k-k'\|_\infty (\|\sigma\|_\infty + \|k\|_\infty + \|k'\|_\infty) \boxed{e^{C_2|\xi|}}.$$

- Inversion is possible yet **severely ill-posed**. Errors grow **exponentially** with frequency.

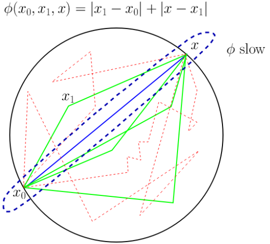
Time-harmonic case [Bal-Jollivet-Langmore-M., CPDE '11]

Goal: to improve the conditioning of the previous problem, go to time-harmonic regime[1] (**frequency-dependent** measurements)

Measurements: $\mathcal{M}^\omega(x_0, x) = \int_{v\cdot\nu_x>0} u_{\delta_{x_0}}(x, v)|v\cdot\nu_x|\; dS(v) \in \mathbb{C}$.

Stationary phase: (case X a ball)

$$\mathcal{M}^\omega(x_0, x) - \mathcal{M}_0^\omega(x_0, x) = \int_X k(x_1) e^{i\omega\phi(x_0,x_1,x)} w(x_0, x_1, x)\; dx_1 + \mathcal{M}_{2+}$$



[1]Similar expansions for small times: [Bal-Jollivet '09]

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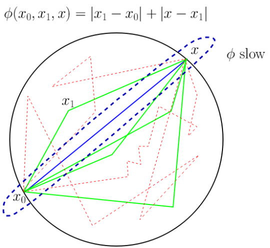
$$\mathcal{M}^\omega(x_0,x) - \mathcal{M}_0^\omega(x_0,x) = \omega^{-\frac{d-1}{2}} \underbrace{R[k](x_0,x)}_{\text{Radon transform}} + o(\omega^{-\frac{d-1}{2}}) \quad \text{in } L^\infty((\partial X)^2).$$



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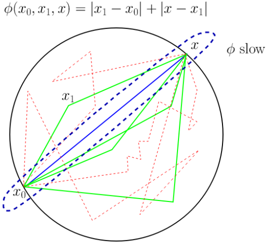


- There is hope to invert for k via an inverse Radon transform.

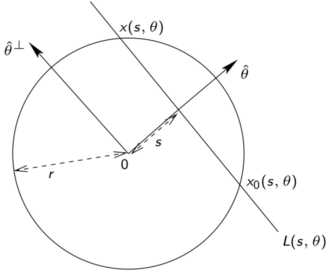
[1] Similar expansions for small times: [Bal-Jollivet '09]

Inversion in two dimensions [Bal-M., SIMA '12]

Problem

Reconstruct $k(x)$ *from* $\{\mathcal{M}^\omega(x_0,x), \quad (x_0,x)\in(\partial X)^2\}$.

1. Parameterize in the **Radon variables** (s,θ),
2. Apply **filtered-backprojection** with cutoff b, call it $R^{-1,b}$
3. Analyse each component using **stationary phase** in (s,θ) **only**.



Resulting estimate (ϕ_b: low-pass filter)

$$\omega^{\frac{1}{2}} R^{-1,b}\mathcal{M}^\omega[k] = \phi_b \star k + \mathcal{O}_{L^\infty}\left(\frac{b^3}{\omega^{\frac{1}{2}}}\log\left(\frac{\omega}{b}\right)\right) \quad \text{(Tradeoff } \omega \leftrightarrow b\text{)}.$$

- **Polynomial stability** in b, instead of **exponential**.

Partial conclusion

Moral of the first story:

1. The reconstruction of k was **severly ill-posed** in the stationary regime.
2. Making the measurements depend on **modulation frequency** by going to the time-harmonic regime allowed to recover **more singularities** ($\approx$ smaller scales) of k.
3. Explicit inversion of **better stability** (hence better resolution) in two dimensions.

SPECT and the inverse source problem

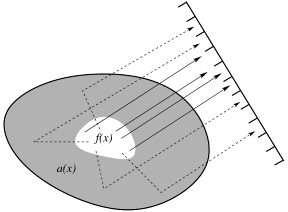
Model: $v \cdot \nabla u + \underbrace{a(x)}_{\text{attenuation}} u = \underbrace{k(x)}_{=Ca(x)} \int u(x,v)\, dv + \underbrace{f(x)}_{\text{source}}, \qquad u|_{\Gamma_-} = 0.$

Measurement:

$u|_{\Gamma_+} = R_a f + \mathcal{M}_1 + \mathcal{M}_{2+}.$

Attenuated Radon transform:

$R_a f(x,v) = \int_{-\infty}^{0} f(x+tv) e^{-\int_t^0 a(x+sv)\, ds}\, dt.$



Problem (The identification problem)

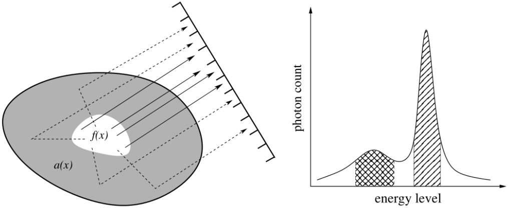
Reconstruct both $a(x)$ and $f(x)$ from $R_a f$.

In this form, the problem is **non-injective** and **unstable**.

[Hertle '88, Boman '93, Solomon '95, Kuchment-Quinto '03, Stefanov '14, Luo-Qian-Stefanov '14]

[Courdurier-M.-Osses-Romero, IP '15]

Idea: Separate ballistic and single scattering from data.



Problem (Identification problem + single scattering data)

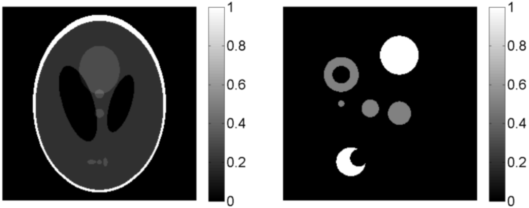
Reconstruct both $a(x)$ and $f(x)$ from $R_a f$ and $\mathcal{M}_1(a, f)$.

- $R_a f(x, v) = \int_{-\infty}^{0} f(x + tv) e^{-\int_t^0 a(x+sv)\, ds}\, dt.$
- $\mathcal{M}_1(a, f)(x, v) = C \int_{-\infty}^{0} a(x + tv) M[a, f](x + tv) e^{-\int_t^0 a(x+sv)\, ds}\, dt.$
- $M[a, f](x, v) = \int_{\mathbb{S}^1} \int_{-\infty}^{0} f(x + tv) e^{-\int_t^0 a(x+sv)\, ds}\, dt\, dv$

Identification problem with single scattering data

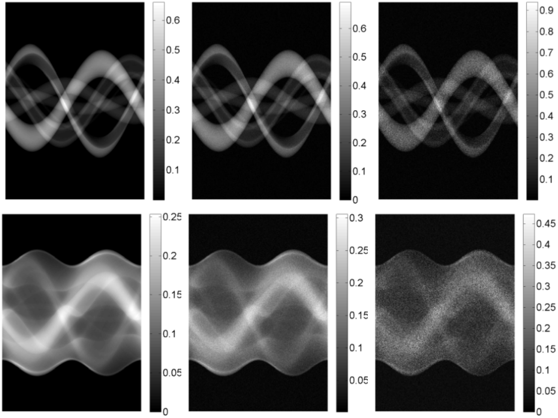
Results in [Courdurier-M.-Osses-Romero, IP '15]:

- Study linearized problem: problem is injective and stable.
- Numerical implementation of Newton's algorithm (+ noise).



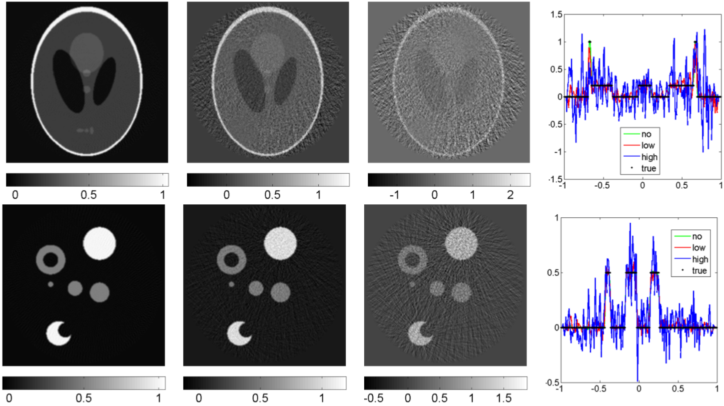
Examples of discontinuous coefficients a (left) and f (right).

Identification problem with single scattering data



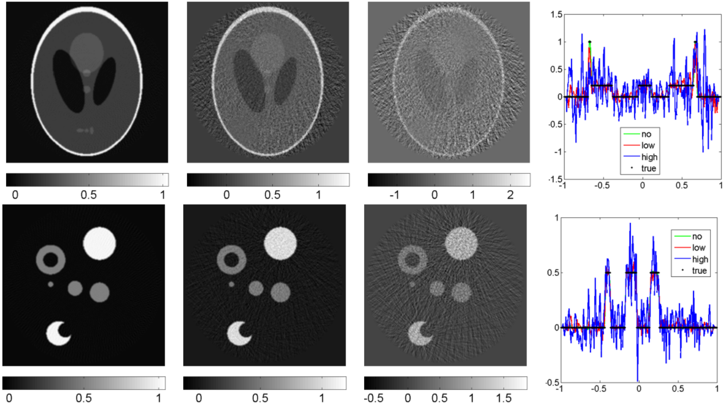
Forward data $\mathcal{A}_0(a, f)$ (top row) and $\mathcal{A}_1(a, f)$ (bottom row)
Left to right: noiseless, low noise, high noise.

Identification problem with single scattering data



Reconstructed a (top row) and f (bottom row) after convergence. Left to right: noiseless, low noise, high noise, cut plots at $\{x = 0\}$.

Identification problem with single scattering data



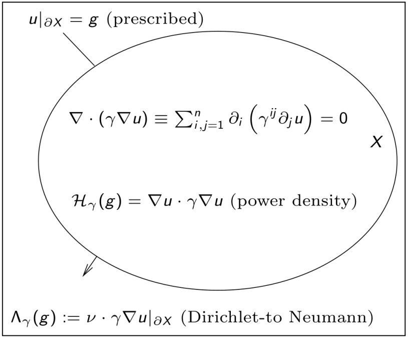
Reconstructed a (top row) and f (bottom row) after convergence. Left to right: noiseless, low noise, high noise, cut plots at $\{x = 0\}$.

- Conclusion: Adding single scattering data stabilized the problem.

Outline

The inverse conductivity problem

Model: $X \subset \mathbb{R}^n$ bounded domain.



γ is *uniformly elliptic*.

Two problems:

- **Calderón's problem:**
Does Λ_γ determine γ uniquely ? stably ?

[Calderón '80]

- **Power density problem:**
Does $\mathcal{H}_\gamma$ determine γ uniquely ? stably ?

Review on Calderón's problem

Problem (Calderón problem)

Determine γ *from the Dirichlet-to-Neuman map* Λ_γ, *where* $f \mapsto \Lambda_\gamma f = \gamma\nu \cdot \nabla u|_{\partial X}$.

The Calderón problem is

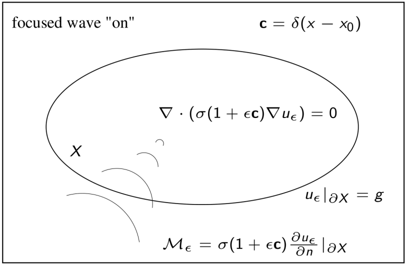
- **injective** in the isotropic case: $\Lambda_\gamma = \Lambda_{\gamma'} \implies \gamma = \gamma'$ [Sylvester-Uhlmann '87, Nachman '88, Novikov '88, Astala-Päivärinta '06, Haberman-Tataru '11, Haberman '14]
- ... yet "**unstable**": The modulus of continuity is logarithmic [Alessandrini '88], which results in low resolution

$$\|\gamma - \gamma'\|_{\mathfrak{X}} \leq C|\ln \|\Lambda_\gamma - \Lambda_{\gamma'}\|_{\mathfrak{Y}}|^{-\delta}.$$

- **non-injective** in the anisotropic case [Lee-Uhlmann '89]

Derivation of power densities - 1/2

By ultrasound modulation



Physical focusing

[Ammari et al. '08]

Synthetic focusing

[Kuchment-Kunyansky '10]

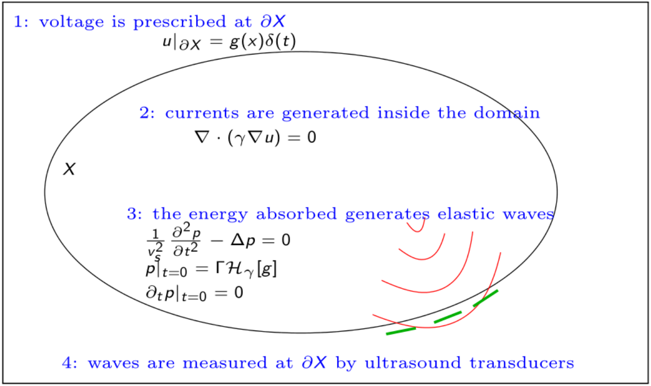
[Bal-Bonnetier-M.-Triki '11]

Small perturbation model:

$\frac{(\mathcal{M}_\epsilon - \mathcal{M}_0)}{\epsilon}$ gives an approximation of $\nabla u_0 \cdot \gamma \nabla u_0$ at x_0.

Derivation of power densities - 2/2

By thermoelastic effects (Impedance-Acoustic CT)



One reconstructs $\Gamma\mathcal{H}_\gamma = \Gamma\nabla u \cdot \gamma\nabla u$ over X (Γ: Grüneisen coefficient)

[Gebauer-Scherzer '09]

Inversion methods

Problem

Reconstruct γ *from* $H_{ij} = \gamma \nabla u_i \cdot \nabla u_j$ *for* $1 \le i,j \le J$.

Injectivity, **stability** and **locality** highly depends on J !

Observation:

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \Leftrightarrow \quad \nabla u \cdot \nabla \log \sigma = -\Delta u.$$

With "enough" measurements, some linearized problems with internal functionals can be formulated in terms of elliptic systems of PDEs (in the sense of Agmon-Douglis-Nirenberg) [Bal, Contemp. Math '14].

Tradeoff: minimal measurements v/s maximal stability and locality of inversion

- J small (e.g., $= 1$): cheap(er) but non-local, less stable
- J large (e.g., $= n$): elliptic system, explicit, local and stable

Inversions from a single functional

Resolution from a single power density - isotropic case:

- Newton-based numerical methods to recover (u, σ) [Ammari et al. '08, Gebauer-Scherzer '09].
- Theoretical work on the Cauchy problem [Bal, Analysis and PDEs '13].

Measurement: $H(x) = \sigma|\nabla u|^2$

- $\nabla \cdot (\sigma \nabla u) = 0$.
- Idea: get u, then σ !
- Problem: instabilities due to hyperbolicity.

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Measurement: $H(x) = \sigma|\nabla u|^2$

- $\nabla \cdot (H \frac{\nabla u}{|\nabla u|^2}) = 0$. "0-laplacian": Hyperbolic PDE.
- Idea: get u, then σ !
- Problem: instabilities due to hyperbolicity.

Isotropic case: formulation

Problem (Isotropic case)

Reconstruct $\sigma \ge c_0 > 0$ *from* $H_{ij}(x) = \sigma \nabla u_i \cdot \nabla u_j$, *where* u_i *solves*

$$\nabla \cdot (\sigma \nabla u_i) = 0, \quad u_i|_{\partial\Omega} = g_i, \quad 1 \le i \le n.$$

Hypothesis:

- $\inf_{\Omega} \det(\nabla u_1, \dots, \nabla u_n) \ge c_1 > 0$ over some $\Omega \subset X$.

Define $F = \frac{1}{2} \nabla \log \sigma$ and $S_i := \sigma^{\frac{1}{2}} \nabla u_i$. The S_i's satisfy

$$\nabla \cdot S_i = -F \cdot S_i, \qquad dS_i = F \wedge S_i, \qquad 1 \le i \le n.$$

The data takes the form $H_{ij} = S_i \cdot S_j$ (Grammian matrix).

Legend: known data, unknown.

Isotropic case: resolution

Using the equations above, derive the **transport equation**

$$F = \frac{1}{n|H|^{\frac{1}{2}}} \left(\nabla (|H|^{\frac{1}{2}} H^{ij}) \cdot S_i \right) S_j.$$

A **first-order quasi-linear system** is then derived for the frame S

$$\boxed{\nabla S_i = \mathcal{S}_i(S, H, dH), \quad 1 \le i \le n}$$

where $\mathcal{S}_i$ is **Lipschitz** w.r.t. $(S_1, \dots, S_n)$.

▶ Overdetermined PDEs, solvable for S, then σ, over $\Omega \subset X$ via ODE's along any (characteristic) curves.

Isotropic case: local stability

Theorem (Uniqueness and Lipschitz stability in $W^{1,\infty}(\Omega)$)

Over $\Omega \subset X$ where the frame assumption is satisfied, σ is **uniquely** *determined up to a constant. Moreover,*

$$\| \log \sigma - \log \sigma' \|_{W^{1,\infty}} \le \varepsilon_0 + C(\|H - H'\|_{W^{1,\infty}})$$

[Capdeboscq et al. '09], [Bal-Bonnetier-M.-Triki, '12], [M.-Bal, IP '12], [M.-Bal, IPI '12], [M.-Bal, CPDE '13]

- **Well-posed** problem (Lipschitz modulus of continuity).
- No loss of derivative/resolution on σ.
- How to achieve $\det(\nabla u_1, \dots, \nabla u_n) > 0$ locally ? globally ?
  [Alessandrini-Nesi 01, Briane-Milton-Nesi 04]

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Anisotropic case: formulation

Problem (Anisotropic case)

Reconstruct the anisotropy $\tilde{\gamma}$ from power densities $\gamma\nabla u_i \cdot \nabla u_j$.

Hypothesis: Start with a "basis" $(u_1, \dots, u_n)$ of solutions, and consider additional solutions with their power densities.

Main ideas:

- An additional solution u_{n+1} decomposes into $\nabla u_{n+1} = \sum_{i=1}^{n} \mu_i \nabla u_i$, where the μ_i's are **known from power densities**.
- Combining this with the PDE's in the problem, one can obtain **orthogonality constraints** on $\tilde{\gamma}$ of the form $Z_p : \tilde{\gamma} = 0$, with Z_p's data matrices.
- If such constraints saturate a hyperplane, $\tilde{\gamma}$ can be reconstructed pointwise and algebraically.

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Stability

Theorem (Uniqueness and stability for $\tilde\gamma$)

Over $\Omega \subset X$ where the hyperplane condition is satisfied, $\tilde\gamma$ is **uniquely**, **explicitly** *determined, with stability*

$$\|\tilde\gamma - \tilde\gamma'\|_{L^\infty(\Omega)} \le C\|H - H'\|_{W^{1,\infty}(X)}.$$

[M.-Bal, IP '12] in 2D. [M.-Bal, CPDE '13] in nD.

- **Explicit reconstruction**. Loss of **one derivative** on $\tilde\gamma$.
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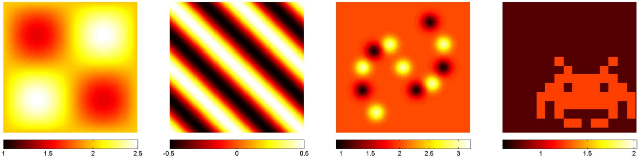
Numerics `[M.-Bal, IP '12]`

Coded in **MatLab** on a cartesian, equispaced grid, using second-order centered **finite differences**.

Decompose $\gamma = \tau\,\tilde{\gamma}$ with $\tilde{\gamma}(\xi,\zeta) = \begin{bmatrix} \xi & \zeta \\ \zeta & \frac{1+\zeta^2}{\xi} \end{bmatrix}$ $(\det \tilde{\gamma} = 1)$.



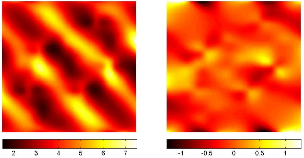
(a) ξ (b) ζ (c) τ smooth (d) τ rough

Compute:

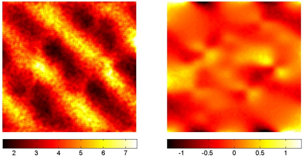
- Solutions (u_1, u_2, u_3) with $(g_1, g_2, g_3)(x,y) = (x+y, y+0.1y^2, -x+y)$
- Data $H_{ij} = \nabla u_i \cdot \gamma \nabla u_j$ with **noise** $H_{noisy} = H.* \ (1 + \frac{\alpha}{100}\ \texttt{random})$.

Numerics - Data H_{ij} and reconstruction of τ

Examples of power densities (H_{11} and H_{12})

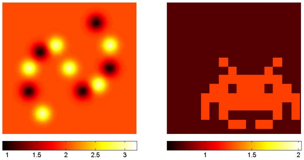


$\alpha = 0\%$

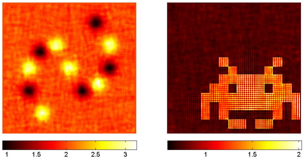


$\alpha = 30\%$

Reconstruction of τ (smooth and rough) with known $\tilde{\gamma}$



true

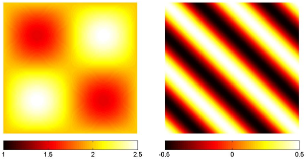


From noisy data ($\alpha = 30\%$)

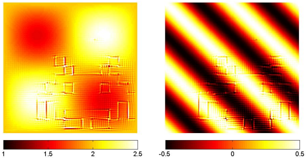
Numerics - reconstruction of $\tilde{\gamma}(\xi,\zeta)$, then τ

Anisotropy reconstruction formula in 2D:

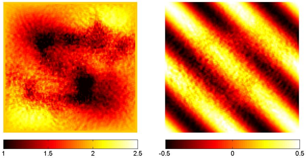
$$\tilde{\gamma} = (JX\cdot Y)^{-1}J(XX^T + YY^T)J, \quad Y = \nabla \log \frac{H_{11}H_{22} - H_{12}^2}{H_{33}H_{44} - H_{34}^2}.$$



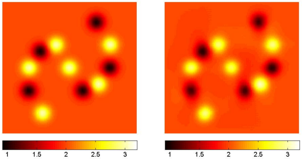
true (ξ,ζ)



with rough τ and $\alpha = 0\%$



with smooth τ and $\alpha = 0.1\%$



τ true and recons. $(\alpha = 0.1\%)$

More inverse problems from internal functionals

Inverse **conductivity** from current densities:

- Model: $\nabla \cdot (\gamma \nabla u) = 0$. Data: $\mathcal{J} = \gamma \nabla u$ inside Ω.
- Application: Current Density Impedance Imaging (coupling EIT and MRI).
- Results:
  - Explicit reconstruction of fully anisotropic tensors. Stability similar as above. [Bal-Guo-M., IPI '14]
  - Numerical simulations. [Bal-Guo-M., SIIMS '14]

Inverse **elasticity** from internal displacement fields:

- Model: $\nabla \cdot (C : (\nabla \mathbf{u} + \nabla \mathbf{u}^T)) = 0$ (system). Data: $\mathbf{u}$ inside Ω.
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- Results: Explicit inversion approaches.
  - Lamé system [Bal-Bellis-Imperiale-M., IP '14]
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Partial conclusion and perspectives

Elliptic inverse problems

- from boundary measurements: **lack of injectivity** (anisotropic case) and **severe instability**.
- from internal functionals motivated by hybrid methods:
  - **Injectivity:** Holds even in fully anisotropic cases
  - **Stability:** Mildly ill-posed instead of severly ill-posed.

Perspectives:

- Regularity reduction.
- More models: reconstruct (γ, σ_a) from $\mathcal{H} = \nabla u \cdot \gamma \nabla u + \sigma_a u^2$, u solves $-\nabla \cdot (\gamma \nabla u) + \sigma_a u = 0$.
- 3D numerical validations,
- Numerical methods: how to solve *overdetermined* PDEs while **enforcing integrability conditions** ?

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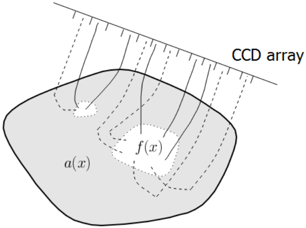
Outline

SPECT with variable refractive index

SPECT (or CT, or OT) can be considered in media with **variable index of refraction**, which acts as a Riemannian metric bending photon trajectories. (Fermat)



Data (before): $R_a f(x,v) = \int_{-\infty}^{0} f(x+tv) e^{-\int_t^0 a(x+sv)\ ds}\ dt.$

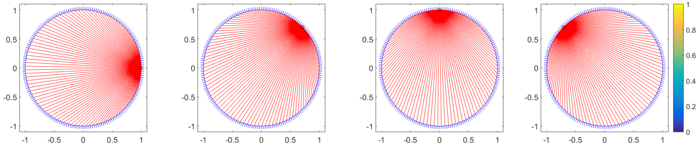
Data (now): $R_a f(x,v) = \int_{-\infty}^{0} f(\gamma_{x,v}(t)) e^{-\int_t^0 a(\gamma_{x,v}(s))\ ds}\ dt.$

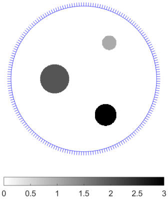
$\gamma_{x,v}(t)$: generated by some flow (geodesic, magnetic, ...)

Incorrect propagation models can lead to significant errors in reconstructions !

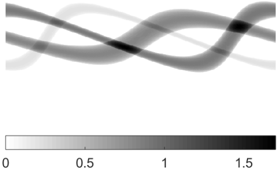
Backprojecting with the wrong geometry

With the **correct** geometry: (0: Euclidean)

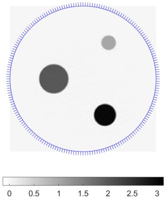




f



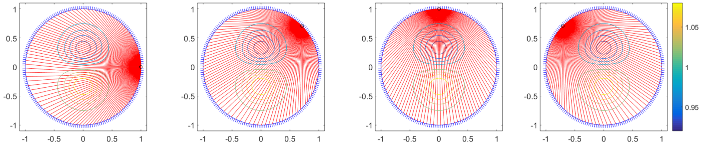
$I_0 f$

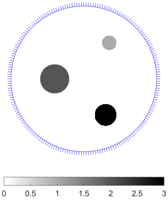


backprojected using $I_0^\sharp$

Backprojecting with the wrong geometry

With an **incorrect** geometry:





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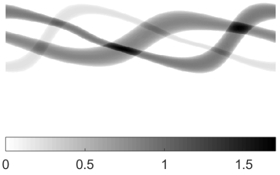
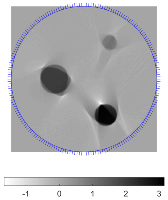


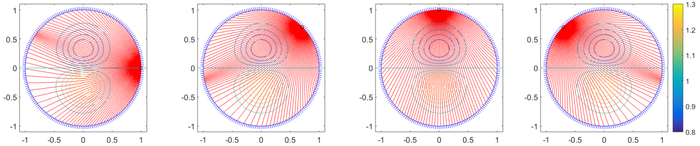
$I_1 f$

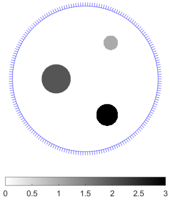


backprojected using $I_0^\sharp$

Backprojecting with the wrong geometry

With a **very incorrect** geometry:





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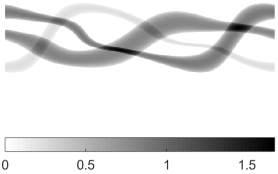
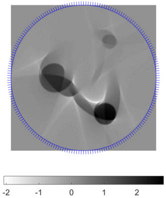


$I_2 f$



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Mathematical formulation

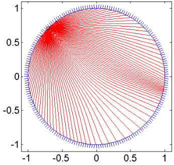
Problem

Given a family of curves Γ, does the collection $\{\int_\gamma f, \gamma \in \Gamma\}$ determine f uniquely ? stably ? How to invert f ?

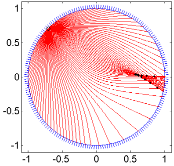
Geometric features of Γ highly impact the answer to these questions.

Non-Euclidean phenomena: curvature, caustics and trapping.

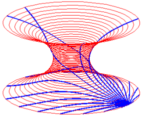
1: "simple"



2: conjugate points



3: trapping



Ongoing and future work

Geodesic ray transforms on Riemannian surfaces with boundary:

1. Numerical implementation of inversions [M., SIIMS '14]
2. Transforms with two types of attenuations [M., IP '15], [M., SIMA '16]
3. Study of caustics [M.-Stefanov-Uhlmann, CMP '15]
4. Study of cases with trapping [Guillarmou-M., preprint '15]

Further questions of interest:

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Thank you !

Research available at:

`www-personal.umich.edu/~monard`