Lecture 8 - Properties of L^1. Modes of convergence

In blue : postponed

§2.3: Integration of complex functions

- ♠ **Thm 2.25:** Suppose $\{f_j\}_j \subset L^1$ satisfies $\sum_{j=1}^\infty \int |f_j| < \infty$. Then $\sum_j f_j$ converges a.e. to $f \in L^1$ and $\int \sum_{j=1}^\infty f_j = \sum_{j=1}^\infty \int f_j$.
- Proof: take a look at $g(x) := \sum_{j=1}^\infty |f_j(x)|$.
- **Thm:** $L^1(\mu)$ is a complete normed space.
- Proof: Thm 2.25 + "a normed space is complete iff every absolute convergent series converges".
- **Thm 2.26:** (i) simple functions are dense in $L^1(\mu)$. (ii) If $X = \mathbb{R}$ and μ is an LS measure, continuous functions are dense in $L^1(\mu)$.
- **Thm 2.27:** Continuity and differentiability of parameter-dependent integrals under domination conditions on the integrand. ($F(t) = \int_X f(t,x)\ d\mu(x)$, $t \in [a,b]$).
- **Thm 2.28:** Let $f\colon [a,b] \to \mathbb{R}$ bounded.
  (i) if f is RI, then f is Lebesgue measurable, integrable, and both integrals agree.
  (ii) f is RI iff the discontinuity set of f is Lebesgue-null.

§2.4: Modes of convergence

- "pointwise" modes: **uniform** $\implies$ **pointwise** $\implies$ **a.e.** but no converse is true.
- L^1 convergence. no implication holds in general. However, DCT says a.e. convergence + a domination condition implies L^1 convergence.
- Examples: (i) $n^{-1}\chi_{(0,n)}$; (ii) $\chi_{(n,n+1)}$; (iii) $n\chi_{[0,1/n]}$; (iv) a sequence that converges nowhere but converges to zero in L^1.
- **Def:** Cauchy in measure; convergence in measure.
- **Thm:** A limit in measure is unique a.e.
- **Thm 2.29:** L^1 convergence implies convergence in measure, but not vice-versa.
- **Thm 2.30:** If $\{f_n\}$ is Cauchy in measure. Then there exists f measurable such that $f_n \to f$ in measure, and a subsequence converges to f a.e.
- In general, a.e. convergence does not imply convergence in measure, but something much stronger holds on finite measure spaces:
- ♠ **Thm 2.33 (Egoroff):** suppose $\mu(X) < \infty$. Let $f_n\colon X \to \mathbb{C}$ measurable and a.e.-convergent to f. Then for every $\epsilon > 0$, there exists $E \subset X$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c.
- Proof: work with $E_n(k) = \cup_{m=n}^\infty \{x : |f_m(x) - f(x)| \ge k^{-1}\}$. $E_n(k)$ decreases in n and $\cap_n E_n(k) = \emptyset$ so $\lim_n \mu(E_n(k)) = 0$.

References

[F] *Real Analysis, modern techniques and their applications*, G. Folland.