Lecture 4 - Borel measures on the real line.

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§1.5: Borel measures on the real line

- **Def:** a Borel measure is a measure $\mu\colon \mathcal{B}_{\mathbb{R}} \to [0,\infty]$.
- A finite Borel measure gives rise to an increasing, right-continuous function $F\colon \mathbb{R} \to \mathbb{R}$ defined by $F(x) := \mu((-\infty, x])$ such that for any $a < b$, $\mu((a,b]) = F(b) - F(a)$.
- Can we turn this process around ? Given F increasing and right-continuous, can one construct a Borel measure μ_F such that $\mu_F((a,b]) = F(b) - F(a)$. (the case $F(x) = x$ is called the Lebesgue measure).
- Construction of a premeasure: define an **h-interval** to be an interval of the form $(a,b]$, (a,∞) or $\emptyset$ for any $-\infty \le a < b < \infty$, and let $\mathcal{A}$ the set of finite disjoint unions of h-intervals. Then $\mathcal{A}$ is an algebra and by Prop. 1.2, $\mathscr{M}(\mathcal{A}) = \mathcal{B}_{\mathbb{R}}$.
- **Prop 1.15:** Let $F\colon \mathbb{R} \to \mathbb{R}$ increasing and right-continuous, and define $\mu_0\left(\cup_{j=1}^n (a_j, b_j]\right) = \sum_{j=1}^n (F(b_j) - F(a_j))$ for any disjoint union of h-intervals, and $\mu_0(\emptyset) = 0$. Then μ_0 is a premeasure on $\mathcal{A}$.
- proof: Check it's well-defined since there is more than one way to represent an element of $\mathcal{A}$. conditional subadditivity: base case of $(a,b] = \cup_{j=1}^\infty (a_j, b_j]$ [. . .]
- **Thm 1.16:** If $F\colon \mathbb{R} \to \mathbb{R}$ is increasing and right-continuous, (i) there exists a unique Borel measure μ_F such that $\mu_F(a,b] = F(b) - F(a)$ for all $a < b$. If G is another function, then $\mu_F = \mu_G$ iff $F - G$ is constant.
  (ii) Conversely, given a Borel measure μ, finite on all Borel sets, then F defined by
  $$F(x) = \begin{cases} \mu((0,x]), & x > 0, \\ 0, & x = 0, \\ -\mu((-x,0]), & x < 0, \end{cases}$$
  is increasing, right-continuous, and $\mu = \mu_F$.
- proof: (i) by prop. 1.15, F induces a premeasure on $\mathcal{A}$, σ-finite since $\mathbb{R} = \cup_{k\in\mathbb{Z}} (k, k+1]$. F induce the same premeasure iff $F - G$ is constant. By thm 1.14, these measures extend uniquely to $\mathcal{B}_{\mathbb{R}}$ and are still equal iff $F - G$ is constant. (ii) monotonicity of F comes from monotonicity of μ. right-continuity comes from continuity of μ from above and below. Finally, $\mu = \mu_F$ on $\mathcal{A}$ thus by uniqueness in Th. 1.14, $\mu = \mu_F$ on $\mathcal{B}_{\mathbb{R}}$.
- **Remarks:**
  - One could also develop the theory using left-cont. functions and right-open intervals.
  - When μ is finite, can also take $F(x) = \mu((-\infty, x])$ (the "cumulative distribution").
  - In fact, one gets more than a Borel measure μ_F but rather a complete measure $\overline{\mu_F}$ on a larger σ-algebra, the **Lebesgue-Stieljes** measure of F.

☐ Properties of the LS measure: let $\mathcal{M}_\mu$ the domain of a fixed LS measure $\mu = \mu_F$. For all $E \in \mathcal{M}_\mu$,

$$\mu(E) = \inf \left\{ \sum_{j=1}^\infty \underbrace{(F(b_j) - F(a_j))}_{\mu((a_j,b_j])} : E \subset \cup_{j=1}^\infty (a_j, b_j] \right\}.$$

- **Lemma 1.17:** $\forall E \in \mathcal{M}_\mu$, $\mu(E) = \inf \left\{ \sum_{j=1}^\infty \mu((a_j, b_j)) : E \subset \cup_{j=1}^\infty (a_j, b_j) \right\}$.
- proof: call $\nu(E)$ the rhs. $\mu(E) \le \nu(E)$: use that for any cover by open intervals, each interval is a countable union of h-intervals. $\nu(E) \le \mu(E) + \varepsilon$: use $(a_j, b_j] \subset (a_j, b_j + \delta_j)$ for δ_j such that $F(b_j + \delta_j) \le F(b_j) + \epsilon 2^{-j}$.
- **Thm 1.18:** For all $E \in \mathcal{M}_\mu$

$$\mu(E) \overset{(i)}{=} \inf \{\mu(U) : E \subset U, U \text{ open}\} \overset{(ii)}{=} \inf \{\mu(K) : K \subset E, K \text{ compact}\}.$$

- proof: (i) is obvious from Lemma 1.17. As for (ii): suppose E bounded; if E is closed, then E is compact thus easy. Otherwise, there is U open such that $U \supset \overline{E} \backslash E$ and $\mu(U) \le \mu(\overline{E} \backslash E) + \epsilon$. Check that $K = \overline{E} \backslash U$ does the trick. If E is unbounded, define $E_j = E \cap (j, j+1]$ and use compacts K_j within $\epsilon 2^{-j}$ measure to E_j.
- **Thm 1.19:** If $E \subset \mathbb{R}$, then (a) $E \in \mathcal{M}_\mu$ iff (b) ($E = V \backslash N_1$ for some G_δ set V and a μ-null set N_1) iff (c) ($E = H \cup N_2$ for some F_σ set H and a μ-null set N_2).
- proof: completeness gives (b) implies (a) and (c) implies (a). Conversely, use Thm 1.18 to produce G_δ and F_σ. Do $\mu(E) < \infty$ and leave the infinite case as exercise.
- Comment: this makes measurable sets relatively simple-looking up to null-sets...

☐ Study of the Lebesgue measure $m = \overline{\mu_F}$ for $F(x) = x$ with domain $\mathcal{L} \supset \mathcal{B}_\mathbb{R}$.

- **Thm 1.21:** m is invariant under translation and scales linearly with dilations.
- Facts:
  - m-null sets include singletons, hence countable sets.
  - Construct an open dense subset of measure ε in $[0, 1]$. The complement is a closed, nowhere dense set of measure $1 - \varepsilon$.
  - Also contain uncountable sets: the Cantor middle-third set: it's compact nowhere dense, totally disconnected with no isolated points, of measure zero and cardinality $\mathfrak{c}$.
  - Cantor-Lebesgue function: for $x = \sum_{j=0}^\infty a_j 3^{-j} \in \mathcal{C}$, let $f(x) := \sum_{j=0}^\infty \frac{a_j}{2} 2^{-j}$, extended by constancy on $[0, 1] \backslash \mathcal{C}$.
  - card $(\mathcal{L})$ = card $(\mathcal{P}(\mathbb{R})) > \mathfrak{c}$ and card $(\mathcal{B}_\mathbb{R}) = \mathfrak{c}$.

References

[F] *Real Analysis, modern techniques and their applications*, G. Folland.