Lecture 18 - Hausdorff measures and fractals

Hausdorff measure and dimension.

- Given a metric space (X, d), we can define a "distance" between two sets by $d(A, B) = \inf_{x\in A, y\in B} d(x, y)$. (note: $d(A, B) = 0$ does not imply $A = B$)
- **Def:** On a metric space (X, d), a **Metric Outer Measure** (MOM) μ^* is an outer measure satisfying: for any two sets A, B, if $d(A, B) > 0$, then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.
- From Carathéodory's thm, MOM's, as outer measures, restrict to measures on the σ-algebra of μ^*-measurable sets. Prop. 11.16 in [F] tells us a bit more: if μ^* is a MOM, then every open set is μ^*-measurable. In particular, **Metric Outer Measures are automatically Borel measures** (by restricting them to $\mathcal{B}_X$).
- Exterior α-dimensional Hausdorff measure: Consider $\mathbb{R}^n$ with its Euclidean distance and for a given set F define $\text{dist}F = \sup\{d(x, y) \colon x, y \in F\}$. For $E \in \mathbb{R}^n$ and $\delta > 0$, let

$$\mathcal{H}_\alpha^\delta := \inf \left\{ \sum_k (\text{diam } F_k)^\alpha, E \subset \bigcup_{k\in\mathbb{N}} F_k,\ \text{diam } F_k \le \delta \right\}.$$

  $\mathcal{H}_\alpha^\delta$ increases as $\delta \to 0$ and thus we can define $m_\alpha^*(E) := \lim_{\delta\to 0} \mathcal{H}_\alpha^\delta(E)$.
- m_α^* is an MOM, hence restricts to $\mathcal{B}_{\mathbb{R}^n}$ as a Borel measure. m_λ^* is translation/rotation invariant, and scales like $m_\alpha^*(\lambda E) = \lambda^\alpha m_\alpha^*(E)$.
- $m_0(E) = \#E$. $m_d(E)$ is proportional to the d-dimensional Lebesgue measure.
- If $m_\alpha^*(E) < \infty$ and $\beta > \alpha$, then $m_\beta^*(E) = 0$. If $m_\alpha^*(E) > 0$ and $\beta < \alpha$, then $m_\beta^*(E) = \infty$. Hence we may define the **Hausdorff dimension** of E.

$$\alpha = \dim E = \sup\{\beta : m_\beta(E) = \infty\} = \inf\{\beta : m_\beta(E) = 0\}.$$

  If $m_\alpha(E) \in (0, \infty)$, we say that E has strict dimension α[1].
- Examples: a segment has strict dimension 1 in any $\mathbb{R}^d$. The k-cube $[0, 1]^k$ has strict dimension k, a non-empty open set in $\mathbb{R}^d$ has strict dimention d. $m_\alpha \equiv 0$ on $\mathbb{R}^d$ if $\alpha > d$.

Given a set E, the task is then two-fold: to find its Hausdorff dimension (the unique $\alpha \ge 0$ such that $m_\alpha(E) \in (0, \infty)$), then to compute $m_\alpha(E)$. The second task is perhaps less relevant (since the scaling property of the α-dimensional measure shows how to get different values by just scaling a given set), so we'll focus on the first problem.

Examples and their Hausdorff dimensions. Cantor set $\mathcal{C}$ ($\frac{\log 2}{\log 3}$), Sierpinski triangle $\mathcal{S}$ ($\frac{\log 3}{\log 2}$), von Koch curve $\mathcal{K}$ ($\frac{\log 4}{\log 3}$), Cantor dust $\mathcal{D}$.

We give some elements of proof of these statements, first focusing on $\mathcal{C}$. We first show that if $\alpha = \frac{\log 2}{\log 3}$, then $m_\alpha(\mathcal{C}) \le 1$. (this implies that the dimension of $\mathcal{C}$ is at most α). Indeed, for $\delta = \frac{1}{3^n}$, we see that $\mathcal{C}$ is covered by 2^n segments of length 3^{-n}. Hence, $\mathcal{H}_\alpha^{3^{-n}}(\mathcal{C}) \le 1$ for all $n \ge 0$, hence sending $n \to \infty$, $m_\alpha(\mathcal{C}) \le 1$.

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[1] We drop the $*$ exponent from m_α^* when thinking of it as a Borel measure.

Proving that $m_\alpha(\mathcal{C}) > 0$ based on Hölder exponents.

- **Def:** For $0 < \alpha \le 1$, f is α-Hölder if $|f(x) - f(y)| \le M|x-y|^\alpha$.
- **Lemma 1:** If E is compact and $f : E \to \mathbb{R}^d$ is γ-Hölder, then for any α, $m_{\alpha/\gamma}(f(E)) \le M^{\alpha/\gamma} m_\alpha(E)$ and $\dim f(E) \le \frac{1}{\gamma}\dim E$.
- proof: think about how diameters of coverings of E are being mapped through f and its Hölder exponents.
- **Lemma 2:** If $f_j : [0,1] \to \mathbb{R}$ is A^j-Lipschitz ($A > 1$) and $|f_{j+1} - f_j|_\infty \le B^{-j}$ ($B > 1$), then f_n has a pointwise limit f which is γ-Hölder, where $\gamma = \log B / \log(AB)$.
- Application: using Lemma 2, one can show that the Cantor-Lebesgue function f is $\frac{\log 2}{\log 3}$-Hölder, and satisfies $f(\mathcal{C}) = [0,1]$. So from lemma 1, we obtain, with $\alpha = \frac{\log 2}{\log 3}$

$$1 = m_1([0,1]) = m_{\alpha/\alpha}(f(\mathcal{C})) \le M m_\alpha(\mathcal{C}),$$

  for some constant $M > 0$, and hence $m_\alpha(\mathcal{C}) > 0$. This completes the proof that $\mathcal{C}$ has strict dimension $\frac{\log 2}{\log 3}$.

Proofs based on self-similarity and the Hausdorff distance.

- A set $F \subset \mathbb{R}^d$ is r-self-similar if there exists r-similarities $S_1, \dots, S_m$, such that

$$F = S_1(F) \cup \cdots \cup S_m(F). \tag{1}$$

- On compact sets of $\mathbb{R}^d$, the Hausdorff distance $\text{dist}(A,B)$ is defined by

$$\text{dist}(A,B) = \inf\{\delta : A \subset B^\delta \text{ and } B \subset A^\delta\}.$$

- dist satisfies: reflexivity, symmetry, triangle inequality, and if $S_1, \dots, S_m$ are r-similarities, upon defining $\tilde{S}(A) = S_1(A) \cup \cdots \cup S_m(A)$, we have $\text{dist}(\tilde{S}(A), \tilde{S}(B)) = r\text{dist}(A,B)$.
- Thm: Given r-similarities $S_1, \dots, S_m$, $\exists\, !F$ compact satisfying (1). (proof: fixed point. First start with B such that $\tilde{S}(B) \subset B$).
- Thm: if the similarities are separated (in that there exists $\mathcal{O}$ open such that $\tilde{S}(\mathcal{O}) \subset \mathcal{O}$ and the sets $S_j(\mathcal{O})$ are disjoint), then $\dim F = \frac{\log m}{\log(1/r)}$.

References

[F] *Real Analysis, modern techniques and their applications*, G. Folland. 1

[SS] *Real Analysis*, Elias M. Stein and Rami Shakarchi, Princeton Lectures in Analysis III.