Lecture 1 - Preliminaries. The exterior measure.

0. Motivation. Some questions:

- how to extend the Riemann integral to more general functions ? Ex: $\chi_{\mathbb{Q}}$

- exists $f_n : [0,1] \to [0,1]$ continuous monotonically decreasing as $n \to \infty$ such that the limiting function is not Riemann integrable. In what context can we rescue the identity $\int_0^1 f(x)\ dx = \lim_{n\to\infty} \int_0^1 f_n(x)\ dx$?

- what is the largest class of functions for which FTC is valid ?

- measure theory: puts volume and integral on a solid mathematical basis. Goal is to create a mapping $E \to m(E)$ generalizing the length, σ-additive, translation invariant. The Lebesgue measure will be a unique such measure, up to restricting it to a class of measurable sets.

1. Preliminaries. Setting: $\mathbb{R}^d$. Define

- point, Euclidean norm, distance.

- set, complement, open, closed, bounded, compact (Heine-Borel property), limit point, isolated point, interior point, closure, boundary, perfect set (= closed with no isolated point).

- rectangles, volume, cubes, almost disjoint (disjoint interiors).

Lemma 1.1: If $R = \bigcup_{k=1}^N R_k$ almost disjoint, then $|R| = \sum_{k=1}^N |R_k|$. (R, R_k rectangles)

Lemma 1.2: If $R \subset \bigcup_{k=1}^N R_k$, then $|R| \le \sum_{k=1}^N |R_k|$.

Thm 1.3: Every open $\mathcal{O} \subset \mathbb{R}$ can be written uniquely as a countable union of open intervals. [gives a way to intuitively measure an open set]

Thm 1.4: Every open $\mathcal{O} \subset \mathbb{R}^d$ can be written as a countable almost disjoint union of closed cubes. [gives a way to intuitively measure an open set]

Define the Cantor set $\mathcal{C}$. Properties: closed, bounded (hence compact); totally disconnected ($\forall x, y \in \mathcal{C},\ \exists z \in (x,y), z \notin \mathcal{C}$); uncountable.

2. The exterior measure. A way to provide a measure for *any* set. For $E \subset \mathbb{R}^d$, define

$$m_*(E) := \inf \left\{ \sum_{j=1}^\infty |Q_j|,\ E \subset \bigcup_{j=1}^\infty Q_j, Q_j \text{ closed cubes} \right\}.$$

Two remarks: (i) finite instead of countable would be insufficient. (ii) can use rectangles or open balls instead.

The definition implies directly **monotonicity** (Obs 1): if $E_1 \subset E_2$, then $m_*(E_1) \subset m_*(E_2)$. Examples:

(1) $m_*(\{x\}) = m_*(\emptyset) = 0$.

(2) $m_*(Q) = |Q|$ is Q is a closed cube.

(3) $m_*(Q) = |Q|$ is Q is an open cube.

(4) $m_*(R) = |R|$ if R is a rectangle.

(5) $m_*(\mathbb{R}^d) = \infty$.

(6) $m_*(\mathcal{C}) = 0$.

References

[SS] *Real Analysis*, Elias M. Stein and Rami Shakarchi, Princeton Lectures in Analysis III.