Lecture 8 - The extended complex plane \( \hat{\mathbb{C}} \), rational functions, Möbius transformations

Material: [G]. [SS, Ch.3 Sec. 3]

The purpose of this lecture is to “compactify” \( \mathbb{C} \) by adjoining to it a point at infinity\(^1 \), and to extend to concept of analyticity there.

Let us first define: a **neighborhood of infinity** \( U \) is the complement of a closed, bounded set. A “basis of neighborhoods” is given by complements of closed disks of the form

\[
U_{z_0, \rho} = \mathbb{C} - \overline{D_\rho(z_0)} = \{ |z - z_0| > \rho \}, \quad z_0 \in \mathbb{C}, \quad \rho > 0.
\]

**Definition 1.** For \( U \) a nbhd of \( \infty \), the function \( f : U \to \mathbb{C} \) has a **limit at infinity** iff there exists \( L \in \mathbb{C} \) such that for every \( \varepsilon > 0 \), there exists \( R > 0 \) such that for any \( |z| > R \), we have \( |f(z) - L| < \varepsilon \).

We write \( \lim_{z \to \infty} f(z) = L \). Equivalently, \( \lim_{z \to \infty} f(z) = L \) if and only if \( \lim_{z \to 0} f \left( \frac{1}{z} \right) = L \).

With this concept, the algebraic limit rules hold in the same way that they hold at finite points when limits are finite.

**Example 1.**
- \( \lim_{z \to \infty} \frac{1}{z} = 0 \).
- \( \lim_{z \to \infty} \frac{z^2 + 1}{(z-1)(3z+7)} = \frac{1}{3} \).
- \( \lim_{z \to \infty} e^z \) does not exist (this is because \( e^z \) has an essential singularity at \( z = 0 \)). A way to prove this is that both sequences \( z_n = \frac{1}{2n\pi i} \) and \( z'_n = \frac{1}{2\pi i(n+1/2)} \) converge to zero, while the sequences \( e^{\frac{1}{zn}} \) and \( e^{\frac{1}{zn}} \) converge to different limits, 1 and 0 respectively.

**Definition 2.** \( \lim_{z \to z_0} f(z) = \infty \) if for every \( M > 0 \), there exists \( \rho > 0 \) such that \( |z - z_0| < \rho \) implies \( |f(z)| > M \). Equivalently, \( \lim_{z \to z_0} f(z) = \infty \) if and only if \( \lim_{z \to z_0} \frac{1}{f(z)} = 0 \).

Combining both definition above, we can say that \( \lim_{z \to \infty} f(z) = \infty \) iff for every \( M > 0 \), there exists \( R > 0 \) such that \( |z| > R \) implies \( |f(z)| > M \).

**Example 2.**
- If \( f \) has a pole of order \( k > 0 \) at \( z_0 \), \( f \) may be written as \( f(z) = \frac{g(z)}{(z-z_0)^k} \) with \( g(z_0) \neq 0 \) and \( g \) analytic near \( z_0 \). Then \( \frac{1}{f(z)} = \frac{(z-z_0)^k}{g(z)} \) so \( \lim_{z \to z_0} \frac{1}{f(z)} = 0 \), i.e., \( \lim_{z \to z_0} f(z) = \infty \).
- For \( f \) any nonconstant polynomial, \( \lim_{z \to \infty} f(z) = \infty \).

**Definition 3.** The extended complex plane (a.k.a. Riemann sphere) is the topological space \( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \), where open sets are generated either by open disks \( D_\rho(z_0) \) or neighborhoods of infinity.

\( \infty \) is now an abstract point which is no different than the other ones. We may study the behavior of a function \( f : \mathbb{C} \to \mathbb{C} \) at \( \infty \) by studying the function \( f(\frac{1}{z}) \) near \( z = 0 \). The question now is: when can such a function be extended as an analytic function \( \hat{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \)?

\(^1\) Adjoining one point to \( \mathbb{C} \) is not the only way to compactify it, this is why this process is called one-point compactification.
Definition 4. \( f : \mathbb{C} \to \mathbb{C} \) is analytic at \( \infty \) iff the function \( w \mapsto f(\frac{1}{w}) \) has a limit in \( \hat{\mathbb{C}} \) at \( w = 0 \) and

(i) if this limit is finite, \( f(\frac{1}{w}) \) is analytic at \( w = 0 \),

(ii) if this limit is \( \infty \), \( \frac{1}{f(\frac{1}{w})} \) is analytic at \( w = 0 \).

Example 3. Show that \( f(z) = \frac{1}{z + 1} \) is analytic at \( \infty \) upon extending it with the value \( -1 \) there.

Definition 5. \( f : \mathbb{C} \to \mathbb{C} \) has a pole of order \( n \) at \( \infty \) if \( f(\frac{1}{w}) \) has a pole of order \( n \) at \( w = 0 \) (we allow \( n = 0 \), which covers the case of a finite limit).

Another way of saying that \( f \) has a pole at \( \infty \) is to say that \( \lim_{z \to \infty} f(z) \) exists in \( \hat{\mathbb{C}} \) (i.e. either as a finite number, or as \( \infty \)).

Theorem 1. \( f \) can be extended into an analytic function \( \tilde{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) if and only if \( f \) is meromorphic on \( \mathbb{C} \) and \( \lim_{z \to \infty} f(z) \) exists, either as a complex number or \( \infty \).

This extension is done as follows: when \( f \) is a meromorphic function on \( \mathbb{C} \) such that \( \lim_{z \to \infty} f(z) \) exists in \( \hat{\mathbb{C}} \), we define \( \tilde{f} : \mathbb{C} \to \hat{\mathbb{C}} \):

- At \( z \in \mathbb{C} \) where \( f \) is analytic, define \( \tilde{f}(z) = f(z) \).
- At \( z \in \mathbb{C} \) where \( f \) has a pole, define \( \tilde{f}(z) = \infty \).
- At \( \infty \), set \( \tilde{f}(\infty) = \lim_{z \to \infty} f(z) \).

Analytic functions on \( \hat{\mathbb{C}} \) define conformal maps near every point where \( f' \neq 0 \).

Theorem 2. The map \( s : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) defined by \( s(z) = \frac{1}{z} \) with \( s(0) = \infty \) and \( s(\infty) = 0 \) is a conformal equivalence of \( \hat{\mathbb{C}} \) onto itself.

Proof. Check that \( s \) is one to one and onto: this is because \( s \circ s = \text{Id} \). \( s \) is analytic at every \( z \neq 0, \infty \). At \( z = \infty \), \( s(\frac{1}{w}) = w \) is analytic at \( w = 0 \) so \( s \) is analytic at \( z = \infty \). At \( z = 0 \), \( \frac{1}{s(z)} = z \) is analytic at \( z = 0 \) so \( s \) is analytic at \( z = 0 \).

Theorem 3. \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) is analytic (or meromorphic) if and only if \( f \) is a rational function.

Proof. We only prove \( (\implies) \), the converse is left to the reader. The first claim is that \( f \) has finitely many poles \( z_1, \ldots, z_n \) on \( \mathbb{C} \) (denote \( (d_1, \ldots, d_n) \) their multiplicities). Indeed, since \( f \) has a limit in \( \hat{\mathbb{C}} \) as \( z \to \infty \), then there exists \( r > 0 \) such that \( w \mapsto f(\frac{1}{w}) \) has no poles on \( D_r(0) \) (you may split cases according to whether \( \infty \) is a pole of \( f \) or not). In particular, \( f \) has no poles on \( \{|z| > \frac{1}{r}, z \in \mathbb{C} \} \).

Then since \( \hat{D}_{\frac{1}{r}}(0) \) is compact, \( f \) has finitely many poles there, so the first claim is proved. We then define \( g(z) := (z - z_1)^{d_1} \cdots (z - z_n)^{d_n} f(z) \), and we now claim that \( g \) is a polynomial. \( g \) has removable singularities at \( z_1, \ldots, z_n \) and is analytic elsewhere, so it’s an entire function. Moreover since \( f \) has a pole at \( \infty \) (of possible order 0), there exists \( d \geq 0 \) such that \( f(\frac{1}{w})w^d \) is bounded near zero, or equivalently, \( f(z)z^{-d} \) is bounded near \( \infty \). This implies that \( g(z)z^{-(d+d_1+\cdots+d_n)} \) is bounded near \( \infty \) and from a previous exercise, since \( g \) is entire, this implies that \( g \) is a polynomial of degree at most \( d + d_1 + \cdots + d_n \). As a conclusion, \( f(z) = \frac{g(z)}{\prod_{j=1}^{n}(z-z_j)^{d_j}} \) is a rational function.
The Riemann sphere  Consider the sphere in $S = \{x_1^2 + x_2^2 + (x_3 - 1/2)^2 = 1/4\} \subset \mathbb{R}^3$ centered at $(0,0,1/2)$, with North Pole $N = (0,0,1)$.

$S$ is sitting on the plane $\{x_3 = 0\} \equiv \mathbb{C}$, and we can define the stereographic projection map, for $M \in S - \{N\}$ to be the unique intersection point between the line $NM$ and the plane $\{x_3 = 0\}$. This map is a homeomorphism: it is bijective, continuous with continuous inverse $\rho^{-1}$. As the neighborhoods of $\infty$ we defined earlier are mapped via $\rho^{-1}$ to neighborhoods of the North Pole $N$, we may extend $\rho : S - \{N\} \to \mathbb{C}$ into a homeomorphism $\tilde{\rho} : S \to \hat{\mathbb{C}}$ by declaring $\tilde{\rho}(N) = \infty$ and $\tilde{\rho}(M) = \rho(M)$ for any $M \in S - \{N\}$.

![Figure 1: Stereographic projection](image)

In that sense, we can assimilate $\hat{\mathbb{C}}$ as a two-dimensional sphere. We will get other (visual) opportunities to see why in the future.

An important property of $\hat{\mathbb{C}}$ is that it is now a compact topological space, unlike $\mathbb{C}$. In particular, every sequence in $\hat{\mathbb{C}}$ has at least one accumulation point. This will imply that discrete subsets of $\hat{\mathbb{C}}$ can only be finite, with consequences on analytic functions $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ (e.g. the “identity” theorem), namely:

**Theorem 4** (Identity theorem on $\hat{\mathbb{C}}$). If $f, g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ are analytic and agree on an infinite set, then they are identically equal.

This implies in particular that such non-trivial functions can only have finitely many zeros and poles.

Linear Fractional (a.k.a. Möbius) transformations

**Definition 6.** A Linear Fractional Transformation is a transformation of the form $h(z) = \frac{az + b}{cz + d}$, with $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$.

Each non-singular $2 \times 2$ complex matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ determines a linear fractional transformation $\phi_A(z) = \frac{az + b}{cz + d}$. A straightforward calculation shows that $\phi_A \circ \phi_B = \phi_{AB}$ and that $\phi_A^{-1} = \phi_{A^{-1}}$, where $AB$ is matrix multiplication and $A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. In other words, the mapping $A \mapsto \phi_A$ is a “group homomorphism”. Most importantly, this property makes it easier to compute the composition of two LFT’s on the fly, by computing the product of the corresponding matrices.
**Example 4.**  
- Affine transformations \( L(z) = az + b \ (c = 0, \ d = 1) \) requires \( a \neq 0 \).
- \( s(z) = \frac{1}{z} \), i.e. \((a, b, c, d) = (0, 1, 1, 0)\).

Note that for every \( \lambda \neq 0 \), the transformations \( h_{a,b,c,d} \) and \( h_{\lambda a, \lambda b, \lambda c, \lambda d} \) define the same transformation. One could add the normalizing condition \( ad - bc = 1 \), though it is sometimes more readable to not normalize.

Let us first notice that every LFT can be written as either (i) \( h = L_1 \) for \( L_1 \) some affine transformation, or (ii) as \( h = L_1 \circ s \circ L_2 \) for \( L_1, L_2 \) two affine transformations. Indeed, if \( c = 0 \), then (i) is satisfied, and if \( c \neq 0 \), we write

\[
h(z) = \frac{az + b}{cz + d} = \frac{cz + d}{cz + d} \cdot \frac{az + b}{cz + d} = \frac{a}{c} + \frac{b - ad}{c} \cdot \frac{1}{cz + d},
\]

so that, upon defining \( L_1(z) = \frac{bc - ad}{c}z + \frac{a}{c} \) and \( L_2(z) = cz + d \), then \( h = L_1 \circ s \circ L_2 \).

**Definition 7.** For \( U \subset \hat{\mathbb{C}} \) an open set, \( h \) is a conformal automorphism of \( U \) (i.e. \( h \in \operatorname{Aut}(U) \)) iff \( h : U \to U \) is conformal and bijective.

**Proposition 5.** For \( U \subset \hat{\mathbb{C}} \) an open set, \( \operatorname{Aut}(U) \) is a group under composition.

*Proof.* The composition of conformal, bijective maps is clearly conformal and bijective. Composition is always associative \(((f \circ g) \circ h = f \circ (g \circ h))\), the identity element is \( Id : U \to U \) and if \( f \in \operatorname{Aut}(U) \), then so it \( f^{-1} \).

**Theorem 6** (Characterization of \( \operatorname{Aut}(\hat{\mathbb{C}}) \)). \( h \in \operatorname{Aut}(\hat{\mathbb{C}}) \) if and only if \( h \) is a linear fractional transformation.

*Proof.* (\( \iff \)) Suppose \( h \) is an LFT. \( h \) is bijective: indeed, if \( h \) is an affine map \( L(z) = az + b \), then it is bijective with inverse \( L^{-1}(w) = \frac{1}{a}(w - b) \); if \( h \) is of the form \( L_1 \circ s \circ L_2 \), i.e. a compound of three invertible transformations, then \( h \) is bijective with inverse \( h^{-1} = L_2^{-1} \circ s \circ L_1^{-1} \). Moreover, \( h \) is conformal: if \( h(z) = L(z) = az + b \), then \( h'(z) = a \neq 0 \) for every \( z \in \mathbb{C} \) so \( h \) is conformal; if \( h = L_2 \circ s \circ L_1, \) \( h \) is conformal as a composition of conformal functions.

(\( \implies \)) Suppose \( f \in \operatorname{Aut}(\hat{\mathbb{C}}) \) (so \( f \) is one to one, onto and conformal).

**First case:** if \( f(\infty) = \infty \), then \( q(z) = s \circ f \circ s \in \operatorname{Aut}(\hat{\mathbb{C}}) \) is such that \( q(0) = 0 \). But \( q'(0) \neq 0 \) since \( q \) is conformal, so \( f \) has a pole of order 1 at \( \infty \). Since \( f \) is one to one on \( \hat{\mathbb{C}} \), \( \infty \) has no other preimage by \( f \), so \( f \) has no pole on \( \mathbb{C} \), i.e. the function \( z^{-1}(f(z) - f(0)) \) is analytic and finite everywhere on \( \hat{\mathbb{C}} \); by Liouville’s theorem, it is constant on \( \mathbb{C} \), then there exists \( c \in \mathbb{C} \) such that \( z^{-1}(f(z) - f(0)) = c \), i.e. \( f(z) = cz + f(0) \).

**Second case:** If \( f(\infty) = k \neq \infty \). Let \( p(z) = \frac{1}{z - k} \) an LFT so \( p \in \operatorname{Aut}(\hat{\mathbb{C}}) \). By the group property, \( p \circ f \in \operatorname{Aut}(\hat{\mathbb{C}}) \) and \( p \circ f(z) = \frac{1}{f(z) - k} \) is such that \( p \circ f(\infty) = \infty \). By the first case, there exists \( a, b \) such that \( az + b = p \circ f(z) = \frac{1}{f(z) - k} \) so that \( f(z) = \frac{1}{az + b} + k \) is an LFT.

As \( \operatorname{Aut}(\hat{\mathbb{C}}) \) can be viewed as the “conformal motions” on the sphere, one may view the action of an LFT on the Riemann sphere dynamically via stereographic projection. See [AR] for a visual example of this.
References

[AR] Möbius transformations revealed, by D. Arnold and J. Rogness, Youtube video


[SS] Complex Analysis, Elias M. Stein and Rami Shakarchi, Princeton Lectures in Analysis II. 1