Lecture 8 - The extended complex plane $\hat{\mathbb{C}}$, rational functions, Möbius transformations

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Material: [G]. [SS, Ch.3 Sec. 3]

The purpose of this lecture is to "compactify" $\mathbb{C}$ by adjoining to it a point at infinity[1], and to extend to concept of analyticity there.

Let us first define: a **neighborhood of infinity** U is the complement of a closed, bounded set. A "basis of neighborhoods" is given by complements of closed disks of the form

$$U_{z_0,\rho} = \mathbb{C} - \overline{D_\rho(z_0)} = \{|z - z_0| > \rho\}, \qquad z_0 \in \mathbb{C}, \qquad \rho > 0.$$

Definition 1. *For U a nbhd of ∞, the function $f : U \to \mathbb{C}$ has a* **limit at infinity** *iff there exists $L \in \mathbb{C}$ such that for every $\varepsilon > 0$, there exists $R > 0$ such that for any $|z| > R$, we have $|f(z) - L| < \varepsilon$. We write $\lim_{z\to\infty} f(z) = L$. Equivalently, $\lim_{z\to\infty} f(z) = L$ if and only if $\lim_{z\to 0} f\left(\frac{1}{z}\right) = L$.*

With this concept, the algebraic limit rules hold in the same way that they hold at finite points when limits are finite.

Example 1.
- $\lim_{z\to\infty} \frac{1}{z} = 0$.
- $\lim_{z\to\infty} \frac{z^2+1}{(z-1)(3z+7)} = \frac{1}{3}$.
- $\lim_{z\to\infty} e^z$ *does not exist (this is because $e^{\frac{1}{z}}$ has an essential singularity at $z = 0$). A way to prove this is that both sequences $z_n = \frac{1}{2n\pi i}$ and $z'_n = \frac{1}{2\pi i(n+1/2)}$ converge to zero, while the sequences $e^{\frac{1}{z_n}}$ and $e^{\frac{1}{z'_n}}$ converge to different limits, 1 and 0 respectively.*

Definition 2. $\lim_{z\to z_0} f(z) = \infty$ *if for every $M > 0$, there exists $\rho > 0$ such that $|z - z_0| < \rho$ implies $|f(z)| > M$. Equivalently, $\lim_{z\to z_0} f(z) = \infty$ if and only if $\lim_{z\to z_0} \frac{1}{f(z)} = 0$.*

Combining both definition above, we can say that $\lim_{z\to\infty} f(z) = \infty$ iff for every $M > 0$, there exists $R > 0$ such that $|z| > R$ implies $|f(z)| > M$.

Example 2.
- *If f has a pole of order $k > 0$ at z_0, f may be written as $f(z) = \frac{g(z)}{(z-z_0)^k}$ with $g(z_0) \neq 0$ and g analytic near z_0. Then $\frac{1}{f(z)} = \frac{(z-z_0)^k}{g(z)}$ so $\lim_{z\to z_0} \frac{1}{f(z)} = 0$, i.e., $\lim_{z\to z_0} f(z) = \infty$.*
- *For f any nonconstant polynomial, $\lim_{z\to\infty} f(z) = \infty$.*

Definition 3. *The extended complex plane (a.k.a. Riemann sphere) is the topological space $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where open sets are generated either by open disks $D_\rho(z_0)$ or neighborhoods of infinity.*

∞ is now an abstract point which is no different than the other ones. We may study the behavior of a function $f : \mathbb{C} \to \mathbb{C}$ at ∞ by studying the function $f(\frac{1}{z})$ near $z = 0$. The question now is: when can such a function be extended as an analytic function $\tilde{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$?

[1] Adjoining one point to $\mathbb{C}$ is not the only way to compactify it, this is why this process is called **one-point** compactification.

Definition 4. *$f : \mathbb{C} \to \mathbb{C}$ is analytic at ∞ iff the function $w \mapsto f(\frac{1}{w})$ has a limit in $\hat{\mathbb{C}}$ at $w = 0$ and*

(i) if this limit is finite, $f(\frac{1}{w})$ is analytic at $w = 0$,

(ii) if this limit is ∞, $\frac{1}{f(\frac{1}{w})}$ is analytic at $w = 0$.

Example 3. *Show that $f(z) = \frac{1-z}{1+z}$ is analytic at ∞ upon extending it with the value -1 there.*

Definition 5. *$f : \mathbb{C} \to \mathbb{C}$ has a pole of order n at ∞ if $f(\frac{1}{w})$ has a pole of order n at $w = 0$ (we allow $n = 0$, which covers the case of a finite limit).*

Another way of saying that f has a pole at ∞ is to say that $\lim_{z\to\infty} f(z)$ exists in $\hat{\mathbb{C}}$ (i.e. either as a finite number, or as ∞).

Theorem 1. *f can be extended into an analytic function $\tilde{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ if and only if f is meromorphic on $\mathbb{C}$ and $\lim_{z\to\infty} f(z)$ exists, either as a complex number or ∞.*

This extension is done as follows : when f is a meromorphic function on $\mathbb{C}$ such that $\lim_{z\to\infty} f(z)$ exists in $\hat{\mathbb{C}}$, we define $\tilde{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$:

- At $z \in \mathbb{C}$ where f is analytic, define $\tilde{f}(z) = f(z)$.
- At $z \in \mathbb{C}$ where f has a pole, define $\tilde{f}(z) = \infty$.
- At ∞, set $\tilde{f}(\infty) = \lim_{z\to\infty} f(z)$.

Analytic functions on $\hat{\mathbb{C}}$ define conformal maps near every point where $f' \neq 0$.

Theorem 2. *The map $s : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by $s(z) = \frac{1}{z}$ with $s(0) = \infty$ and $s(\infty) = 0$ is a conformal equivalence of $\hat{\mathbb{C}}$ onto itself.*

Proof. Check that s is one to one and onto: this is because $s \circ s = Id$. s is analytic at every $z \neq 0, \infty$. At $z = \infty$, $s(\frac{1}{w}) = w$ is analytic at $w = 0$ so s is analytic at $z = \infty$. At $z = 0$, $\frac{1}{s(z)} = z$ is analytic at $z = 0$ so s is analytic at $z = 0$. □

Theorem 3. *$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is analytic (or meromorphic) if and only if f is a rational function.*

Proof. We only prove ($\Longrightarrow$), the converse is left to the reader. The first claim is that f has finitely many poles $z_1, \dots, z_n$ on $\mathbb{C}$ (denote $(d_1, \dots, d_n)$ their multiplicities). Indeed, since f has a limit in $\hat{\mathbb{C}}$ as $z \to \infty$, then there exists $r > 0$ such that $w \mapsto f(\frac{1}{w})$ has not poles on $\dot{D}_r(0)$ (you may split cases according to whether ∞ is a pole of f or not). In particular, f has no poles on $\{|z| > \frac{1}{r}, z \in \mathbb{C}\}$. Then since $\overline{D_{\frac{1}{r}}(0)}$ is compact, f has finitely many poles there, so the first claim is proved. We then define $g(z) := (z - z_1)^{d_1} \cdots (z - z_n)^{d_n} f(z)$, and we now claim that g is a polynomial. g has removable singularities at $z_1, \dots, z_n$ and is analytic elsewhere, so it's an entire function. Moreover since f has a pole at ∞ (of possible order 0), there exists $d \geq 0$ such that $f(\frac{1}{w})w^d$ is bounded near zero, or equivalently, $f(z)z^{-d}$ is bounded near ∞. This implies that $g(z)z^{-(d+d_1+\cdots+d_n)}$ is bounded near ∞ and from a previous exercise, since g is entire, this implies that g is a polynomial of degree at most $d + d_1 + \cdots + d_n$. As a conclusion, $f(z) = \frac{g(z)}{\prod_{j=1}^{n}(z-z_j)^{d_j}}$ is a rational function. □

The Riemann sphere Consider the sphere in $S = \{x_1^2 + x_2^2 + (x_3 - 1/2)^2 = 1/4\} \subset \mathbb{R}^3$ centered at $(0, 0, 1/2)$, with North Pole $N = (0, 0, 1)$.

S is sitting on the plane $\{x_3 = 0\} \equiv \mathbb{C}$, and we can define the stereographic projection map, for $M \in S - \{N\}$ to be the unique intersection point between the line NM and the plane $\{x_3 = 0\}$. This map is a homeomorphism: it is bijective, continuous with continuous inverse ρ^{-1}. As the neighborhoods of ∞ we defined earlier are mapped via ρ^{-1} to neighborhoods of the North Pole N, we may extend $\rho : S - \{N\} \to \mathbb{C}$ into a homeomorphism $\tilde{\rho} : S \to \hat{\mathbb{C}}$ by declaring $\tilde{\rho}(N) = \infty$ and $\tilde{\rho}(M) = \rho(M)$ for any $M \in S - \{N\}$.

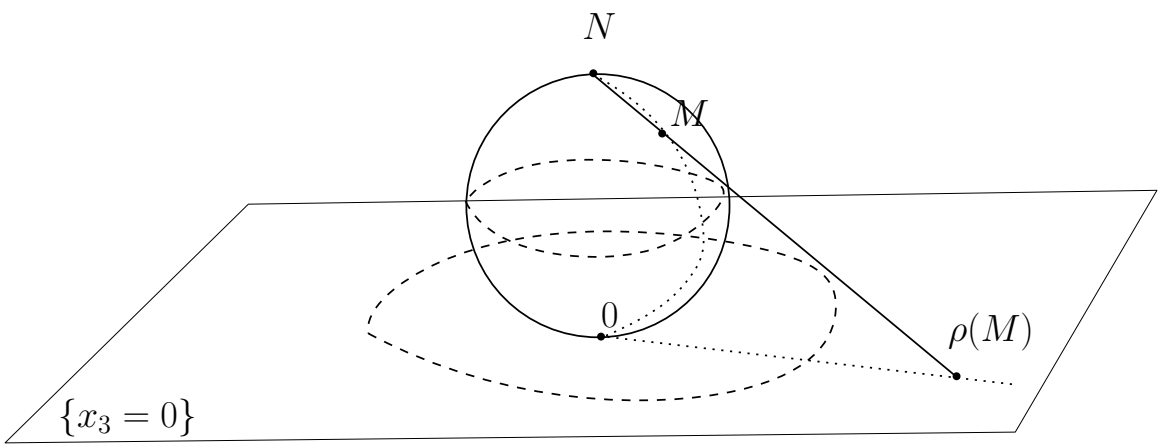

Figure 1: Stereographic projection

In that sense, we can assimilate $\hat{\mathbb{C}}$ as a two-dimensional sphere. We will get other (visual) opportunities to see why in the future.

An important property of $\hat{\mathbb{C}}$ is that it is now a compact topological space, unlike $\mathbb{C}$. In particular, every sequence in $\hat{\mathbb{C}}$ has at least one accumulation point. This will imply that discrete subsets of $\hat{\mathbb{C}}$ can only be finite, with consequences on analytic functions $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ (e.g. the "identity" theorem), namely:

Theorem 4 (Identity theorem on $\hat{\mathbb{C}}$)**.** *If $f, g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ are analytic and agree on an infinite set, then they are identically equal.*

This implies in particular that such non-trivial functions can only have finitely many zeros and poles.

Linear Fractional (a.k.a. Möbius) transformations

Definition 6. *A* **Linear Fractional Transformation** *is a transformation of the form $h(z) = \frac{az+b}{cz+d}$, with $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$.*

Each non-singular 2×2 complex matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ determines a linear fractional transformation $\phi_A(z) = \frac{az+b}{cz+d}$. A straighforward calculation shows that $\phi_A \circ \phi_B = \phi_{AB}$ and that $\phi_A^{-1} = \phi_{A^{-1}}$, where AB is matrix multiplication and $A^{-1} = \frac{1}{ad-bc}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. In other words, the mapping $A \mapsto \phi_A$ is a "group homomorphism". Most importantly, this property makes it easier to compute the composition of two LFT's on the fly, by computing the product of the corresponding matrices.

Example 4.
- *Affine transformations* $L(z) = az + b$ *(*$c = 0$*,* $d = 1$*) requires* $a \neq 0$*.*
- $s(z) = \frac{1}{z}$*, i.e.* $(a, b, c, d) = (0, 1, 1, 0)$*.*

Note that for every $\lambda \neq 0$, the transformations $h_{a,b,c,d}$ and $h_{\lambda a,\lambda b,\lambda c,\lambda d}$ define the same transformation. One could add the normalizing condition $ad - bc = 1$, though it is sometimes more readable to not normalize.

Let us first notice that every LFT can be written as either (i) $h = L_1$ for L_1 some affine transformation, or (ii) as $h = L_1 \circ s \circ L_2$ for L_1, L_2 two affine transformations. Indeed, if $c = 0$, then (i) is satisfied, and if $c \neq 0$, we write

$$h(z) = \frac{az + b}{cz + d} = \frac{\frac{a}{c}(cz + d) + b - \frac{ad}{c}}{cz + d} = \frac{a}{c} + \frac{bc - ad}{c}\frac{1}{cz + d},$$

so that, upon defining $L_1(z) = \frac{bc-ad}{c}z + \frac{a}{c}$ and $L_2(z) = cz + d$, then $h = L_1 \circ s \circ L_2$.

Definition 7. *For* $U \subset \hat{\mathbb{C}}$ *an open set,* h *is a conformal automorphism of* U *(i.e.* $h \in Aut\ (U)$*) iff* $h : U \to U$ *is conformal and bijective.*

Proposition 5. *For* $U \subset \hat{\mathbb{C}}$ *an open set,* $Aut\ (U)$ *is a group under composition.*

Proof. The composition of conformal, bijective maps is clearly conformal and bijective. Composition is always associative ($(f \circ g) \circ h = f \circ (g \circ h)$), the identity element is $Id : U \to U$ and if $f \in$ Aut (U), then so it f^{-1}. □

Theorem 6 (Characterization of Aut $(\hat{\mathbb{C}})$)**.** $h \in Aut\ (\hat{\mathbb{C}})$ *if and only if* h *is a linear fractional transformation.*

Proof. ($\Longleftarrow$) Suppose h is an LFT. h is bijective: indeed, if h is an affine map $L(z) = az + b$, then it is bijective with inverse $L^{-1}(w) = \frac{1}{a}(w - b)$; if h is of the form $L_1 \circ s \circ L_2$, i.e. a coumpound of three invertible transformations, then h is bijective with inverse $h^{-1} = L_2^{-1} \circ s \circ L_1^{-1}$. Moreover, h is conformal: if $h(z) = L(z) = az + b$, then $h'(z) = a \neq 0$ for every $z \in \mathbb{C}$ so h is conformal; if $h = L_2 \circ s \circ L_1$, h is conformal as a composition of conformal functions.

($\Longrightarrow$) Suppose $f \in$ Aut $(\hat{\mathbb{C}})$ (so f is one to one, onto and conformal).
<u>First case:</u> if $f(\infty) = \infty$, then $q(z) = s \circ f \circ s \in$ Aut $(\hat{\mathbb{C}})$ is such that $q(0) = 0$. But $q'(0) \neq 0$ since q is conformal, so f has a pole of order 1 at ∞. Since f is one to one on $\hat{\mathbb{C}}$, ∞ has no other preimage by f, so f has no pole on $\mathbb{C}$, i.e. the function $z^{-1}(f(z) - f(0))$ is analytic and finite everywhere on $\hat{\mathbb{C}}$: by Liouville's theorem, it is constant on $\mathbb{C}$, then there exists $c \in \mathbb{C}$ such that $z^{-1}(f(z) - f(0)) = c$, i.e. $f(z) = cz + f(0)$.
<u>Second case:</u> If $f(\infty) = k \neq \infty$. Let $p(z) = \frac{1}{z-k}$ an LFT so $p \in$ Aut $(\hat{\mathbb{C}})$. By the group property, $p \circ f \in$ Aut $(\hat{\mathbb{C}})$ and $p \circ f(z) = \frac{1}{f(z)-k}$ is such that $p \circ f(\infty) = \infty$. By the first case, there exists a, b such that $az + b = p \circ f(z) = \frac{1}{f(z)-k}$ so that $f(z) = \frac{1}{az+b} + k$ is an LFT. □

As Aut $(\hat{\mathbb{C}})$ can be viewed as the "conformal motions" on the sphere, one may view the action of an LFT on the Riemann sphere dynamically via stereographic projection. See [AR] for a visual example of this.

References

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