**Lecture 20 - Introduction to complex dynamics - 3/3: Mandelbrot and friends**

Outline:

- Recall critical points and behavior of functions nearby.
- Motivate the proof on connectedness of polynomials for Julia sets.
- Mandelbrot set.
- The cubic family.

## Zoology of Fatou sets

Here, we will merely state facts about the types of behavior one can expect on the Fatou set. We have seen an example before, which was the basin of attraction of a fixed points.

Let $\Omega$ a connected component of the Fatou set. Since the family $\{F^n\}$ has normally convergent subsequences there (call accumulation function its limit), one may ask the question: how many accumulation functions does the sequence $\{F^n\}_n$ have ? If it is a finite number, we may expect to be in the basin of attraction of an attracting/superattracting or neutral periodic cycle. In the latter case, such connected components are referred to as *parabolic cycles*. For example, it can be shown that a neighborhood of a neutral fixed point is split into an even number of sectors, where orbits alternatively converge or diverge away from that fixed point. Finally, if the number of accumulation functions of the sequence $\{F^n\}_n$ is infinite (think for instance of the sequence of iterates of the map $F(z) = e^{2i\pi\alpha}z$ with $\alpha$ irrational), depending on the connectedness of $\Omega$, it may lead to a so-called *Siegel disk* or a *Herman ring*. The interested reader may find more detail on this topic in [DK], Article "Julia sets" by Linda Keen.

## Quasi self-similarity of Julia sets

The fractal quality of Julia sets is formalized by the concept of quasi-self-similarity. Self-similarity occurs when a set is a dilated, isometric copy of one of its proper subsets, see e.g. the Von Koch curve or the Cantor set. In our case, Julia sets are rarely self-similar because the "smaller copies" are slightly distorted, though this distortion remains bounded in some sense. One may thus think of the concept of quasi-self-similarity as a self-similarity up to small distortions.

**Definition 1.** *$\phi : U \to V$ is a $K$-*quasi-isometry *$(K \ge 1)$ if for every $x, y \in U$, we have*

$$\frac{1}{K}|x-y| \le |\phi(x) - \phi(y)| \le K|x-y|.$$

*If $K = 1$, then $|\phi(x) - \phi(y)| = |x - y|$ and $\phi$ is an isometry (e.g. in Euclidean space, a composition of rotation and translation).*

**Theorem 1** (Sullivan)**.** *If $F$ is a rational, expanding map, then the Julia set of $F$ is such that there exists $K \ge 1$ and $r_0 > 0$ such that for every $z \in \mathcal{J}$ and every $0 < r < r_0$, the set $\mathcal{J} \cap D_r(z)$, dilated by a factor $\frac{1}{r}$, is $K$-quasi-isometric to the whole of $\mathcal{J}$.*

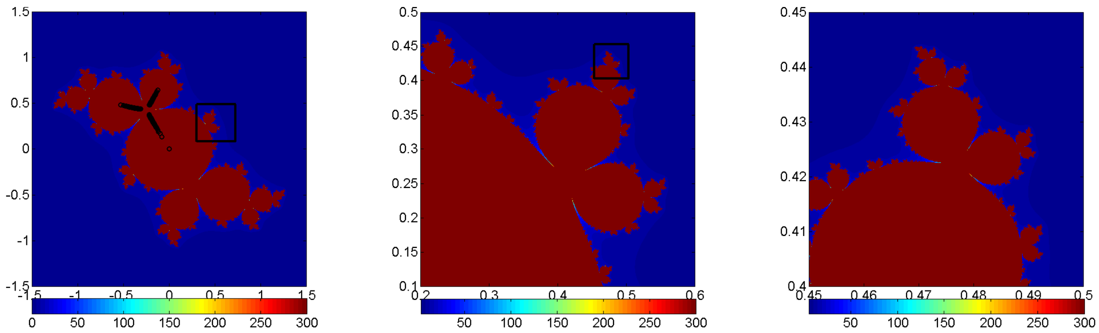


Figure 1: Examples of zooms into a Julia set (zoom-ins from l. to r.), illustrating quasi-self-similarity

## Numerical representation of Julia sets

Let us present two methods in order to visualize Julia sets.

Method 1: Inverse iteration method

The first method is based on an earlier result that for any $w \in \mathcal{J}$, the set $\bigcup_{n\in\mathbb{N}} P_c^{-n}(w)$ is dense in $\mathcal{J}$. The method therefore consists in (i) picking a point $w_0$ in $\mathcal{J}$ and (ii) computing successively the preimages $P_c^{-n}(w_0)$.

**Example 1.** *Fix $c \in \mathbb{C}$ and define $P_c(z) = z^2 + c$. For this function, the two steps described above go as follows: Step (i): We know that $P_c(z) = z^2 + c$ has fixed points $z_\pm = \frac{1}{2} \pm \sqrt{\left(\frac{1}{4} - c\right)}$ and that $z_+$ is always repelling, therefore $z_+ \in \mathcal{J}$.*
*Step (ii): Computing preimages of a given $w$ consists in solving for $z$ the equation $z^2+c = w$, which obviously has two solutions $z = \pm\sqrt{w-c}$, and these are the two preimages of $w$. So by induction, we can establish that*

$$P_c^{-(n+1)}(w_0) = \{\pm\sqrt{w-c}, \quad w \in P_c^{-n}(w_0)\}.$$

*We can see that $P_c^{-n}(w_0)$ has exactly $2^n$ distinct elements.*

Method 2: Boundary scanning method

The main drawback of the previous method is that it tries to capture the Julia set spot on, whose chaotic nature does not pair well with the finite-accurary computations of computers.

The second method is based on the observation that for a function of the form $P_c(z) = z^2 + c$, $\infty$ is a superattracting point, so there is $R > 0$ such that $U := \hat{\mathbb{C}}\backslash\overline{D_R(0)} \subset A_{P_c}(\infty)$ and moreover, $\mathcal{J}(P_c) = \partial A_{P_c}(\infty) \subset \overline{D_R(0)}$. A quick estimate also shows that $R = \max(2, |c|)$ works. The method then goes as follows:
Fix a large integer $N_{max}$ (say 500).

- $(i)$ Discretize a fine enough grid of the square $[-R, R] \times [-R, R]$.
- $(ii)$ For each gridpoint $z$, compute its orbit for $N_{max}$ steps and set $n(z)$ to be either the smallest integer where $|P_c^n(z)| > R$, or $N_{max}$ if the orbit never exitted $\overline{D_2(0)}$.
- $(iii)$ Display $n(z)$.

Instead of focusing on the boundary itself, we focus on visualizing $A_{P_c}(\infty)$ as best we can, by computing at each point of a given grid in how many steps the orbit of the point arrives in $U$. $n(z)$ is sometimes referred to as the *escape rate*. One may notice that in some cases, a whole region of the plane is colored with the value $N_{max}$, this is because in some cases, the Fatou set of $P_c$ is made of two components, one of which is bounded by the Julia set. We sometimes refer to this region as the *filled-in Julia set*. We will see that the condition of whether there exists another bounded component of the Fatou set of $P_c$ is the very criterion which defines the Mandelbrot set.

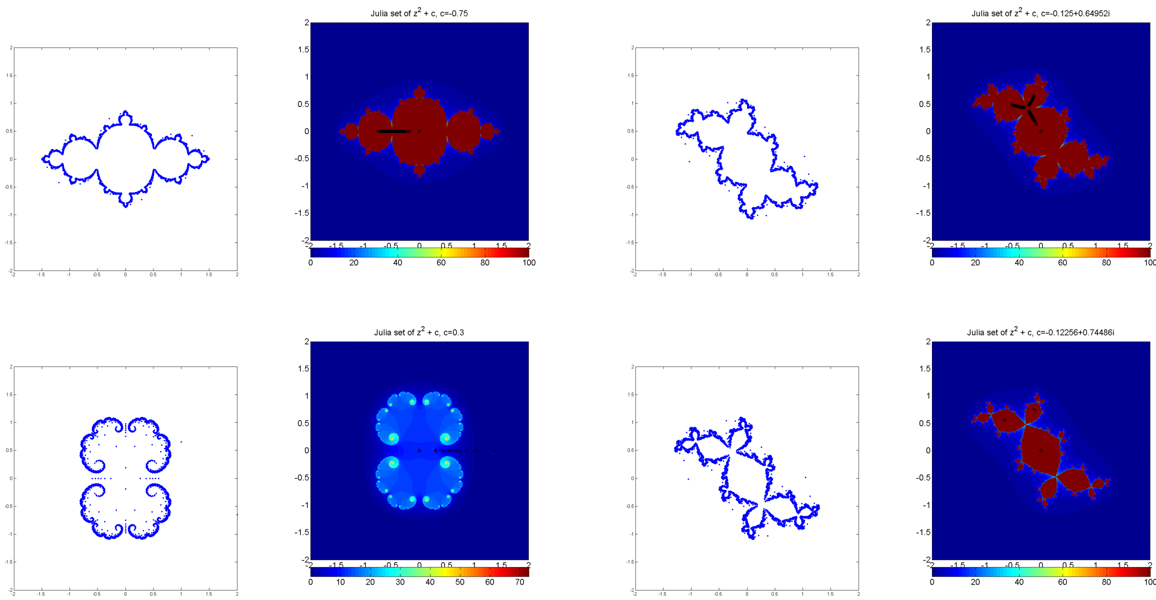


Figure 2: Examples of Julia sets using both visualization methods on the iteration of polynomials of the form $P_c(z) = z^2 + c$.

Figure 2 shows examples of Julia sets for four different values of $c$. We see that the first method, though much less expensive, lacks accuracy because the function that is iterated over is $\sqrt{z}$ instead of $z$ (the first method relies on inverse orbits while the second relies on forward orbits). This explains why in the outcome of method 1, some points lie outside the "main cloud". Except in the bottom-left example, this is NOT to be expected, as one can show that the Julia set is in fact connected.

Another advantage of Method 2 over Method 1 is that, for polynomials of degree $> 2$, it becomes tricky to compute preimages by polynomials of high degrees. This can still be done with relative ease on the function $F(z) = z - \frac{z^3-1}{3z^2}$ as seen on Fig. 3 (the Newton map of $z \mapsto z^3 - 1$), though in most cases, it seems much more convenient to use the boundary scanning method, which is easy to implement for polynomials of arbitrary order...

## Critical points and the Mandelbrot set

It is now time to stress the importance of the orbits of critical points in complex dynamics.

**Theorem 2** (Fatou, 1905)**.** *Every attracting cycle for a rational function attracts at least one critical point.*

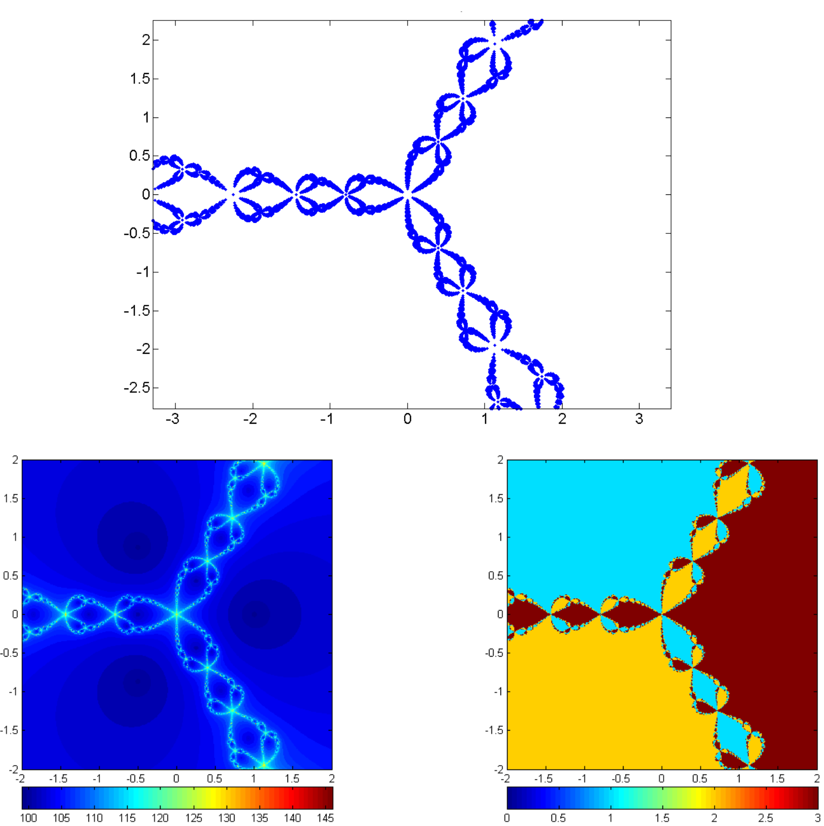


Figure 3: Top: Julia set of Newton map of $f(z) = z^3 - 1$ via inverse iterates. Bottom: same Julia set obtained via the boundary scanning method.

This means that if $F = \frac{P}{Q}$ is a rational function with $P, Q$ polynomials of degree at most $d$, the equation $F' = 0$ consists in finding the roots of $P'Q - PQ'$, a polynomial of degree at most $2d-1$, so $F$ has at most $2d-1$ critical points and can only have at most $2d-1$ attracting cycles.

In the case of the family $P_c(z) = z^2 + c$, we have $P_c'(z) = 2z$ regardless of $c$, and $z = 0$ is the only critical point, so $P_c$ has at most one attracting cycle.

Another important theorem is the following. For $F$ a rational map, denote $K_F$ the set of seeds with bounded orbit, and $\Omega_F$ the set of critical points of $F$.

**Theorem 3** (Fatou, Julia)**.** *If $F$ is a polynomial, then we have:*
*(i) $\Omega_F \subset K_F$ if and only if $\mathcal{J}_F$ is connected.*
*(ii) if $\Omega_F \cap K_F = \emptyset$, then $\mathcal{J}_F$ is a Cantor set.*

Since $\Omega_F = \{0\}$ when $F = P_c$, these are the only cases to consider. The $c$-plane is then split into two sets where either $0 \in K_{P_c}$ or $0 \notin K_{P_c}$.

**Definition 2.** *The* Mandelbrot set $\mathcal{M}$ *is the set of $c$ in the parameter plane such that case (i) in Theorem 3 occurs.*

**Remark 1.** *The Mandelbrot set is a set in the c-plane, or the 'parameter' plane, whereas the Julia set is a set of the 'dynamical' plane $z$. Each value of $c \in \mathbb{C}$ leads to a different Julia set.*

In other words, $c \notin \mathcal{M}$ if and only if the orbit of 0 under $P_c$, that is to say,

$$\text{the sequence} \qquad 0,\ c,\ c^2+c,\ (c^2+c)^2+c,\ \dots \qquad \text{is unbounded.}$$

It is this observation which motivates the computer graphics representing $M$, which one can write along the same lines as the second method for representing Julia sets presented above. It is easy to show that if $|c| > 2$, then the sequence $\{P_c^n(0)\}_n$ always diverges, so that $\mathcal{M} \subset D_2(0)$.

Method for visualizing $\mathcal{M}$: Fix a large integer $N_{max}$ (say 500).

$(i)$ Discretize the square $[-2, 2] \times [-2, 2]$ finely enough.

$(ii)$ For each gridpoint $c$, compute terms of the sequence $P_c^n(0)$ until they exit $D_2(0)$ or until $n$ reaches $N_{max}$. Define $n(c)$ as either the smallest integer less than $N_{max}$ such that $|P_c^n(0)| > 2$, or $N_{max}$ if the orbit of $c$ remains in $D_2(0)$ until $N_{max}$.

$(iii)$ Visualize $n(c)$.

**Some facts about the Mandelbrot set.** As interior points of the Mandelbrot set are such that the orbit of 0 is bounded, we may wonder what its behavior is, in particular, how many accumulation points the sequence $\{P_c^n(0)\}_n$ has. When 0 is attracted to a $k$-cycle, that sequence has exactly $k$ accumulation points. In general, we define $H$ a *hyperbolic component* of $\mathcal{M}$ a component of $\mathcal{M}$ where the orbit of the point 0 is attracted to a cycle of fixed period.

One can show by calculation that 0 is attracted to a fixed point (or 1-cycle) if and only if $c$ belongs to the largest subset denoted $W_0$ on Fig. 4, so-called the *main cardioid*. We denote this set $W_0$, of boundary equation $\rho_{W_0}(t) = \frac{e^{2\pi i t}}{2} - \frac{e^{4\pi i t}}{4}$, $t \in [0, 1]$.

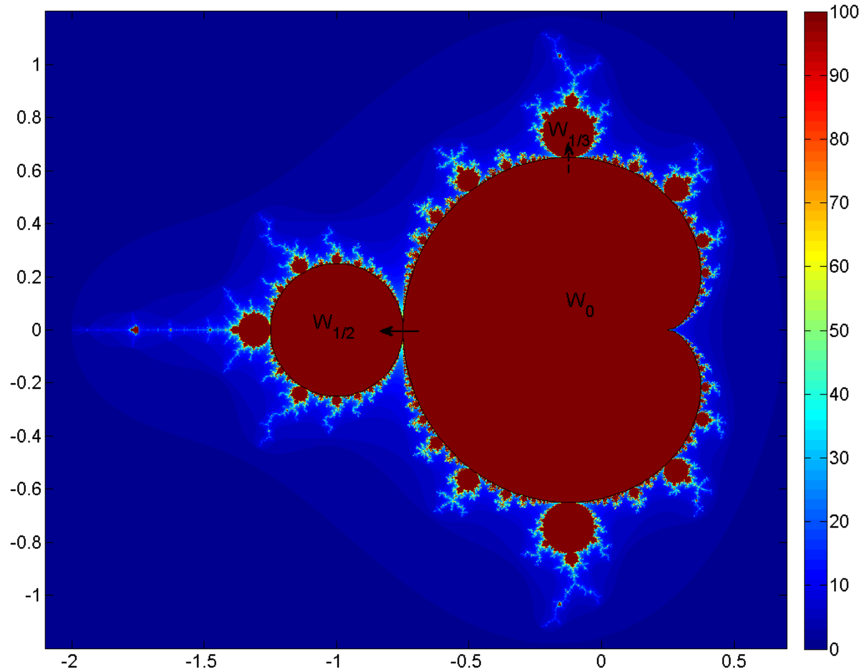


Figure 4: The Mandelbrot set $\mathcal{M}$. As $c$ follows the plain arrow, the system undergoes a perio-doubling bifurcation. As $c$ follows the dashed arrow, the system undergoes a period-tripling bifurcation.

The parameter value $c = \rho_{W_0}\left(\frac{1}{2}\right) = -\frac{3}{4}$ is where a period-doubling bifurcation occurs, as we have seen in the lectures on one-dimensional dynamics (which corresponds to when $c$ is on the real-axis), what we analyzed about the family $F_c(x) = x^2 + c$, in particular the bifurcation diagram, tells us a great deal here !), see Figure 5. When $c$ is slightly to the left of $\frac{-3}{4}$, we are in another hyperbolic component (so-called $W_{\frac{1}{2}}$) where the point 0 is attracted to a 2-cycle. The multiplier of this 2-cycle has precise expression $4(c+1)$ so that this multiplier has modulus less than 1 if and only if $c \in D_{\frac{1}{4}}(-1)$, in particular we have $W_{1/2} = D_{\frac{1}{4}}(-1)$. At the boundary of components such as $W_0$ or $W_{\frac{1}{2}}$, there are typically neutral cycles happening, since these are the bordeline cases where a cycle passes from attracting to repelling and another passes from repelling to attracting.

In the same way that a bifurcation occurs at $c = \gamma_{W_0}(\frac{1}{2})$, one can show that a period $q$-doubling bifurcation occurs at each point $\gamma_{W_0}\left(\frac{p}{q}\right)$: when $c$ changes from being inside the big cardioid to inside the hyperbolic component $W_{\frac{p}{q}}$, the point 0 changes from being attracted to a fixed point to being attracted to a $q$-cycle.

At points $\gamma_{W_0}(t)$ where $t$ is irrational, in the same way that the sequence $\{e^{i\alpha n}\}_n$ is dense in the unit circle when $\alpha$ is irrational, the orbit of 0 seems to have infinitely many accumulation points inside the Julia set. This is the case where the Julia set is so-called a *Siegel disk*, see example Figure 6, right.

There are many more exciting facts about the Mandelbrot set (including things that yet remain to be proved !) and the interested reader can take a look at the article "The Mandelbrot set" by Bodil Branner in [DK] for more interesting details.

**Additional properties.** Against expectations and despite the fact that the Mandelbrot set looks

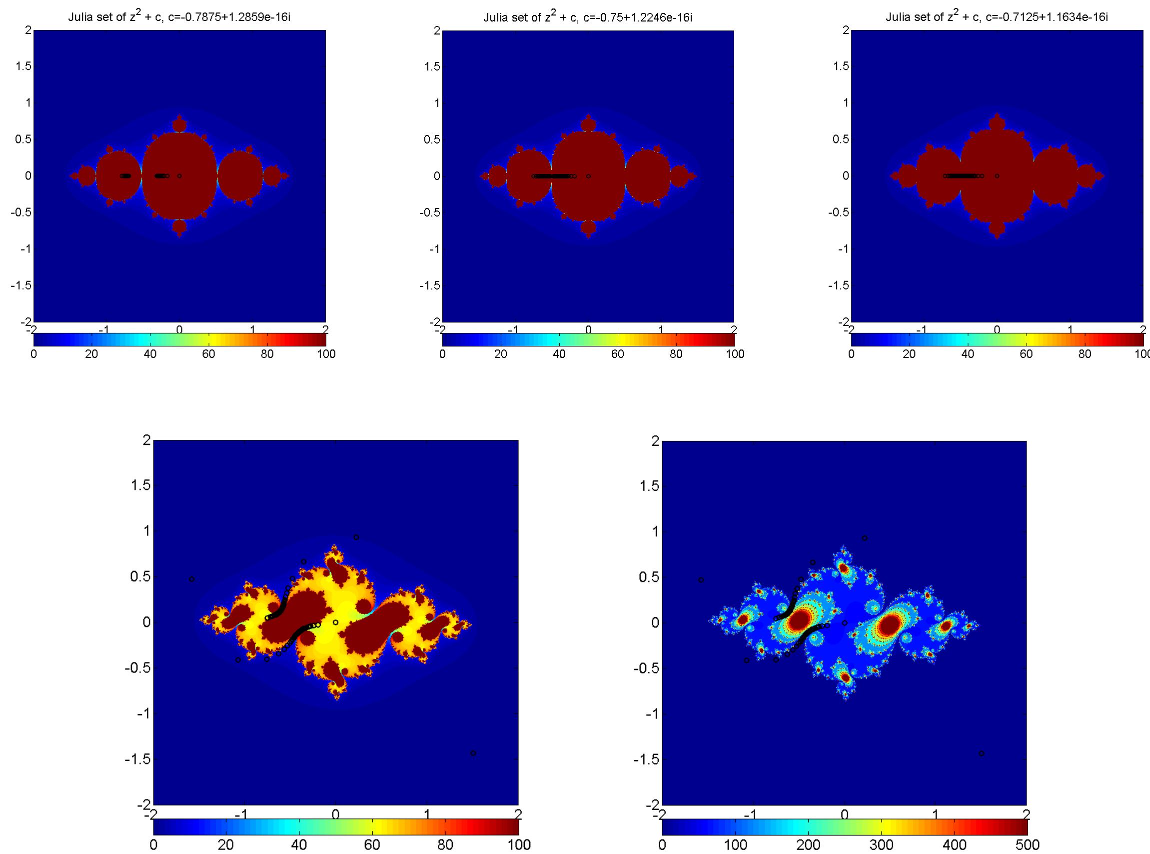


Figure 5: Near the point $c = \frac{-3}{4}$. Top, left to right: Julia set of $P_c$ for $c$ real, taking values $c < \frac{-3}{4}$, $c = \frac{-3}{4}$ and $c > \frac{-3}{4}$. We see on the left that the orbit of 0 converges to a 2-cycle while on the right, it converges to a single point. Bottom: $c = -\frac{3}{4} + 0.1i$, outside the Mandelbrot set. The Julia set is completely disconnected and all orbits outside $\mathcal{J}$ eventually diverge to $\infty$. However they can take arbitrarily long to do so, this si why we must increase $N_{max}$ in order to visualize additional structure (left is $N_{max} = 100$, right is $N_{max} = 500$). In either picture, the constant red regions no longer means that the corresponding seeds have bounded orbits; rather, $N_{max}$ was too small to see these seeds reach the region $\hat{\mathbb{C}} \backslash D_2(0)$.

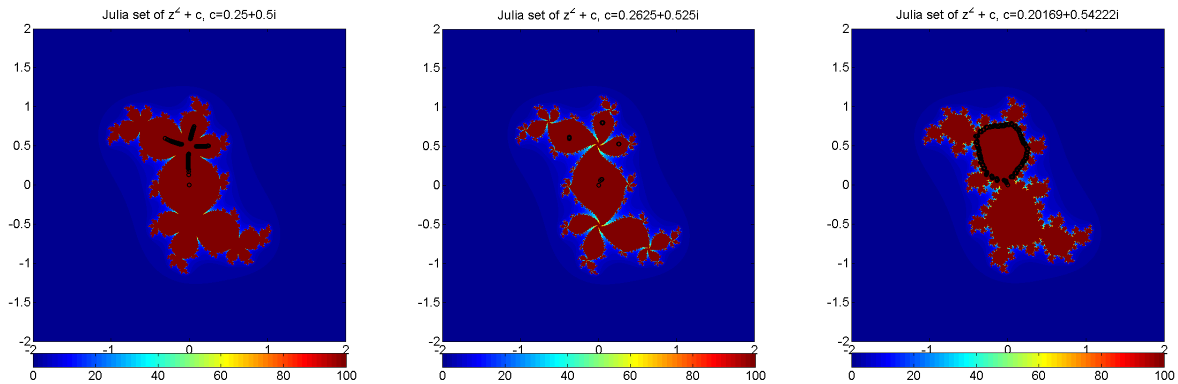


Figure 6: Left-middle: near the period 4-doubling bifurcation at $c = \gamma_{W_0}\left(\frac{1}{4}\right)$. Right: Example of a Siegel disk, where $c = \gamma_{W_0}(t)$ for some irrational $t$.

like an agglomerate of cardiods and disks of various sizes, it turns out that it is *not* (quasi-)self-similar, unlike Julia sets. The smaller copies each have their own set of decorations which differ from the other copies. On the other hand, sets in the shape of a Mandelbrot set appear in many other one-parameter families of complex dynamics, including higher-order polynomials, a property that gives the Mandelbrot set the attribute to be *universal*.

Let us give an example of this last fact. For $\rho \in \mathbb{C}$, consider the family of cubics $P_\rho(z) = z(z-1)(z-\rho)$ and define the Newton map $N_\rho(z) = z - \frac{P_\rho(z)}{P_\rho'(z)}$, the map to be iterated over. We know that $N_\rho$ has three superattracting fixed points at 0, 1 and $\rho$, and since they are superattracting, this precisely means that they are also critical, and are attracted by their respective basins. However, there is a fourth critical point[1] with value $z_c = \frac{1+\rho}{3}$, and one may wonder into what basin of attraction its orbit falls. For each value of $\rho$, we can then assign a color to the pixel $\rho$ depending on what basin of attraction it falls into, though we may notice that some values are in fact attracted to a fourth attracting cycle. What's more is that such regions in the $\rho$-plane take the form of Mandelbrot sets, for which the previous discussion adapts: the hyperbolic component of this Mandelbrot set in which $\rho$ lies will determine the period of the additional cycle to which $z_c$ is attracted.

There are infinitely many such Mandelbrot sets inside the $\rho$-plane, and whenever $\rho$ is in one of them, the Julia set $\mathcal{J}(N_\rho)$ in the $z$-plane contains small features reminiscent of the Julia set of some quadratic polynomial !

**Zooming into Mandelbrot.** Although it is not self-similar, the Mandelbrot set is still very complicated at every scale, and one may want to zoom into it at arbitrarily fine scales. Using a finer and finer grid requires increasing the precision of the computations, which in turn become computationally more intensive. See [Zoom] for an example of deep zoom into the Mandelbrot set, of a factor $10^{228}$, taking months to render. For the record the biggest number in physical ranges is in the order of $10^{42}$ (size of a proton to the universe).

This is where the separation lies between fractals in mathematics and fractals in physics: scale-invariant principles in physics lead to fractal structures which are self-similar at a few scales, then this self-similarity stops due to a change in the governing laws of nature at some limiting scale. Their mathematical counterparts are, however, defined for arbitrarily small or large scales.

[1] When $P_\rho$ has simple roots, the equation $N_\rho'(z) = 0$ is equivalent to $P_\rho(z) = 0$ or $P_\rho''(z) = 0$. This second equation gives us the last critical point.

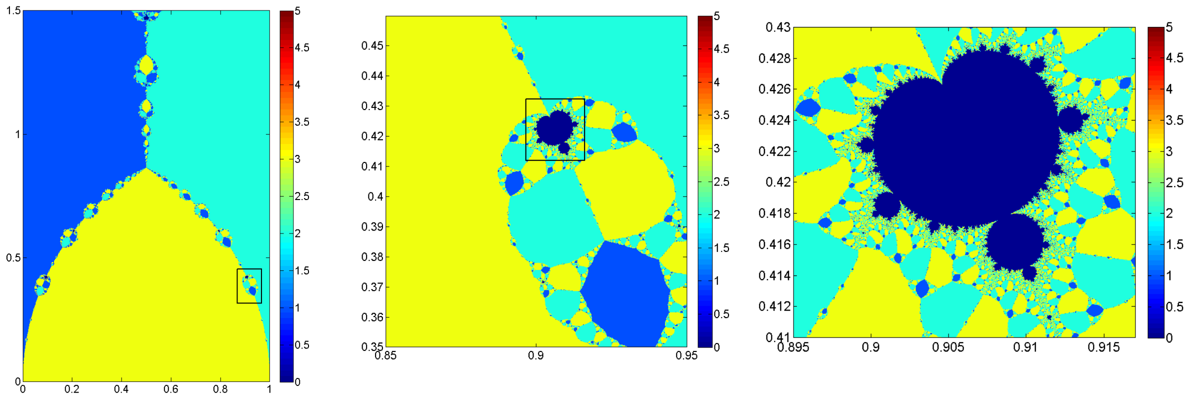


Figure 7: Left to right: zooms into the $\rho$-plane of the family of Newton maps of $P_\rho(z) = z(z-1)(z-\rho)$. Pixels in yellow/turquoise/light blue corresponds to cases where $z_c = \frac{1+\rho}{3}$ is attracted to the fixed point $\rho/1/0$, respectively. The dark blue pixels are when there is a fourth attracting cycle, to which $z_c$ is attracted.

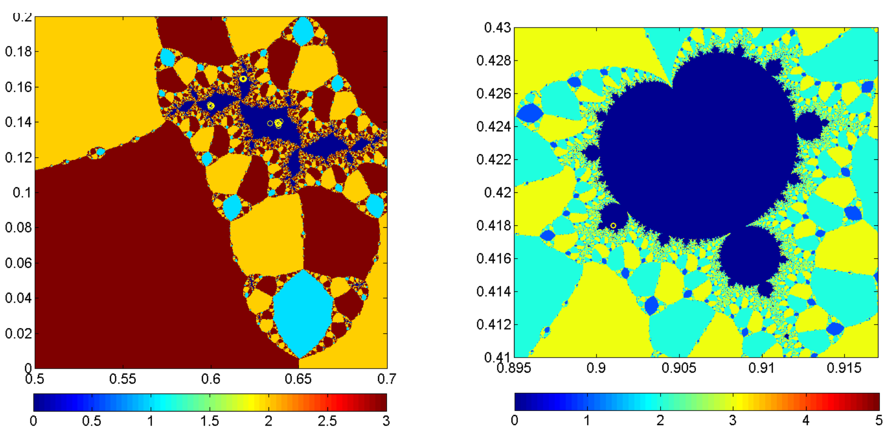


Figure 8: Right: $\rho$-plane with a parameter value $\rho$ inside the hyperbolic component of an attracting 3-cycle. Left: the corresponding $z$-plane, partitioned into the three basins of attraction of the roots of $P_\rho$, as well as the additional basin of attraction of a 3-cycle, whose boundary resembles the Julia set of some quadratic polynomial $z \mapsto z^2 + c$ (compare to Fig. 2, bottom).

## References

[A] *A History of complex dynamics. From Schröder to Fatou and Julia*, by Daniel S. Alexander. Aspects of Mathematics, Vieweg 1994.

[DK] *Chaos and Fractals, the mathematics behind the computer graphics.* Editors: Devaney and Keen. 1, 6

[G] *Complex Analysis*, Theodore W. Gamelin. Undergraduate Texts in Mathematics, Springer.

[Zoom] "Deepest Mandelbrot set zoom animation ever" (video)

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