Lecture 19 - Introduction to complex dynamics - 2/3: Julia and Fatou

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Outline:

- Quadratic family: the regime $c < -2$. Chaotic map (formal definition).
- The quadratic map $z \mapsto z^2$.
- Global conjugacies. Newton's algo for quadratic polynomials. Full answer based on the study of $F(z) = z^2$.
- Julia and Fatou sets. General facts on invariant sets.

The quadratic family - continued

Recall that for $c \in [-2, 1/4]$, the mapping $F_c(x) = x^2 + c$ has two real fixed points $x_\pm(c) = \frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$ and the interval $[-x_+(c), x_+(c)]$ is stable under F_c.

The regime $c < -2$. Chaotic behavior. We would now like to study what happens when $c < -2$, for which the definition of Cantor set above comes in handy. When $c < -2$, something new and dramatic happens: the interval $[-x_+, x_+]$ is no longer stable under F because the bottom of the parabola is poking through the box $[-x_+, x_+] \times [-x_+, x_+]$. This will cause some orbits that initially started inside $[-x_+, x_+]$ to actually converge to $+\infty$. The question is now, do all orbits eventually converge to $+\infty$? The method to study that is to follow backwards the orbits that go to $+\infty$ and cover all possible scenarios. Graphically, we can find out by successive elimination of all orbits converging to $+\infty$ that the seeds of orbits remaining in $[-x_+, x_+]$ form at most a set that is homeomorphic to the Cantor middle-third set. Call this set Λ. At this point, we will merely state facts about what actually happens, though proving anything would be beyond the scope of this introduction.

There indeed exists a set $\Lambda \subset [-x_+, x_+]$ that is homeomorphic to a Cantor set, and such that $F_c(\Lambda) \subset \Lambda$, so we can consider the dynamical system restricted to Λ. The dynamics on Λ is called *chaotic* in the sense of the definition below, which requires putting an appropriate topology on Λ (the topology of open intervals is now inappropriate since we know that Λ does not contain an open interval and looks like a collection of discrete points). This is done by defining a certain *metric function* $d(x, y)$ on Λ, which in turn allows to reintroduces the notion of *open set* and *neighborhood* (i.e. neighborhoods would be "spanned" by sets of the form $D_\delta(x) = \{y \in \Lambda,\ d(x, y) < \delta\}$). For a special choice of such a distance function, we can show that $F_c : \Lambda \to \Lambda$ is chaotic in the sense of the definition below.

Definition 1. *Let (Λ, d) a metric space with distance function $d(x, y)$ and $F : \Lambda \to \Lambda$. F is called* chaotic *if the following conditions are satisfied*

(i) F is topologically transitive*, this means that for every U, V open sets in Λ, there exists k such that $F^k(U) \cap V \neq \emptyset$.*

(ii) F *has* sensitive dependence to initial conditions, *this means that there exists* $\delta > 0$ *such that for every* $x \in \Lambda$ *and every neighborhood* V *of* x *there exists* $y \in V$ *and* $n \in \mathbb{N}$ *such that* $d(F^n(x), F^n(y)) \geq \delta$. [1]

(iii) *Periodic points of* F *are dense in* Λ.

Point (i) means that no matter the sets U, V, there will always be a seed in U whose orbit will visit V. Point (ii) means that there will always be points arbitrarily close to one another whose orbits will be separated by an absolute distance.

As Edward Lorentz describes it, "Chaos is when the present determines the future, but the approximate present does not approximately determine the future." It is essentially a description of lack of predictability, despite the fact that the system is completely deterministic.

Dynamics under $z \mapsto z^2$

Now that an example of one-dimensional dynamics has showed us a variety of dynamical behaviors, we are ready to explore two-dimensional dynamics.

The function to be iterated will now be a function of a complex variable. In particular, we will use a *rational* function $F(z) = \frac{P(z)}{Q(z)}$, where P and Q are polynomials. Then F can be extended into an analytic function $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, and we must study the dynamical system

$$z_0 \in \hat{\mathbb{C}}, \quad z_{n+1} = F(z_n), \quad n \in \mathbb{N}.$$

As in the one-dimensional case, we define $z^\star$ to be a *k-periodic* point of F if $F^k(z^\star) = z^\star$ and for every $1 \leq j < k$, $F^j(z^\star) \neq z^\star$. The orbit of a k-periodic point is periodic with period k, i.e. it is of the form

$$z^\star, z_1, \dots, z_{k-1}, z^\star, z_1, \dots, z_{k-1}, etc \dots$$

and it is called a k-cycle. Of course, the case $k = 1$ covers the case of fixed points of F. For a k-periodic point $z^\star$, we call its *multiplier* $\lambda = (F^k)'(z^\star)$, whose modulus allows us to tell whether the point $z^\star$ is *superattracting* ($\lambda = 0$), *attracting* ($|\lambda| < 1$), *neutral* ($|\lambda| = 1$) or *repelling* ($|\lambda| > 1$). The point ∞ can also be a fixed point, in which case the multiplier to consider is $\lambda = \frac{1}{F'(\infty)}$.

Finally, for a fixed point $z^\star$ of F, we define the *basin of attraction* of $z^\star$ as the set of seeds in $\hat{\mathbb{C}}$ whose orbits converge to $z^\star$, that is,

$$A_F(z^\star) = \{z \in \hat{\mathbb{C}} : \lim_{n\to\infty} F^n(z) = z^\star\}.$$

Remark 1. *Because when a limit is unique whenever it exists, the basins of attractions of two distinct fixed points cannot overlap. So each basin of attraction covers a certain part of* $\hat{\mathbb{C}}$. *The question is, do they cover the whole sphere ? If not, what happens to the dynamics outside the basins of attraction ? The answer is, chaotic behavior. Let us now study an example in detail.*

[1] Equivalently, there exists $\delta > 0$ such that for every $x \in \Lambda$ and every $\varepsilon > 0$, there exists $y \in \Lambda$ and $n \in \mathbb{N}$ such that $d(x, y) < \varepsilon$ and $d(F^n(x), F^n(y)) > \delta$.

An example of a chaotic function. Let us study the dynamical system associated with the function $F(z) = z^2$. In this case, we can actually compute the iterates explicitely and arrive at $F^n(z) = z^{2^n}$.
We first look for the fixed points of F, which after solving $F(z) = z$ yields $z \in \{0, 1\}$. We also need to consider ∞ and we notice also that $F(\infty) = \infty$ so the fixed points are $z \in \{0, 1, \infty\}$. The multipliers of each of these points are $(F'(0), F'(1), \frac{1}{F'(\infty)}) = (0, 2, 0)$, respectively, so 0 and ∞ are superattracting while 1 is repelling.

From these basic observations, we realize that 0 and ∞ will be "competing" to attract orbits. The interface between these basins of attraction will be made of points whose orbits should not be expected to converge anywhere and will trigger chaotic dynamical behavior.

Further straightforward observations lead to the following partitioning of the Riemann sphere into three sets, each of which is stable by F: $\hat{\mathbb{C}} = U_1 \cup U_2 \cup U_3$, where $U_1 = \mathbb{D}$, $U_2 = \partial\mathbb{D}$ and $U_3 = \hat{\mathbb{C}}\backslash\overline{\mathbb{D}}$ and for $1 \le j \le 3$, $F(U_j) \subset U_j$. This is because due to the form of F, we have that $|z| < 1$ implies $|F(z)| < 1$, $|z| = 1$ implies $|F(z)| = 1$ and $|z| > 1$ implies $|F(z)| > 1$. Moreover, if $|z_1| < 1$, then $\lim_{n\to\infty} z^{2^n} = 0$ and if $|z| > 1$, $\lim_{n\to\infty} z^{2^n} = \infty$. So we have completely characterized the basins of attraction of 0 and ∞, that is $A_F(0) = \mathbb{D}$ and $A_F(\infty) = \hat{\mathbb{C}}\backslash\overline{\mathbb{D}}$. It remains to study the behavior of F on the unit circle $\partial\mathbb{D}$.

Theorem 1. *$F(z) = z^2$ is chaotic as a mapping from $\partial\mathbb{D}$ to itself.*

Note that when discussing about chaos in the real quadratic family, we had trouble understanding what it meant to put an *ad hoc* topology on Cantor sets. Here we can work with the usual distance on the unit circle, i.e. for two points on the unit circle $x = e^{i\alpha}$ and $y = e^{i\beta}$, we will denote

$$d(x, y) = \mod (\alpha - \beta, 2\pi).$$

For this distance, the "basic" open sets are open arc circles of the form $\{e^{i\theta}, \theta \in (\alpha - \varepsilon, \alpha + \varepsilon)\}$, this particular one being an ε-neighborhood of $e^{i\alpha}$.

Proof of Theorem 1. We need to show that F is (i) topologically transtive, (ii) has sensitive dependence on initial conditions and (iii) its periodic points are dense in $\partial\mathbb{D}$.

Proof of (i). We need to show that for U, V two open sets, there exists $k > 0$ such that $F^k(U) \cap V \neq \emptyset$. Pick U an open set, then it contains a "basic" open set of the form $U_0 = \{e^{i\theta}, \theta \in (\alpha - \varepsilon, \alpha + \varepsilon)\}$. For any $k > 0$, we have

$$F^k(U_0) = \{e^{i2^k\theta}, \theta \in (\alpha - \varepsilon, \alpha + \varepsilon)\} = \{e^{i\theta}, \theta \in (2^k\alpha - 2^k\varepsilon, 2^k\alpha + 2^k\varepsilon)\}.$$

So once k is such that $2^k\varepsilon > \pi$, $F^k(U_0)$ contains the whole unit circle, so no matter what V is, $F^k(U) \cap V \neq \emptyset$. F is therefore topologically transitive.

Proof of (ii). We need to prove that there exists $\delta > 0$ such that for every $x \in \partial\mathbb{D}$, for every $\varepsilon > 0$, there is y and $k > 0$ such that $d(x, y) < \varepsilon$ and $d(F^k(x), F^k(y)) > \delta$. Let us show that $\delta = \frac{\pi}{2}$ does the trick. Pick $x = e^{i\alpha}$ and for $\varepsilon > 0$ small, $y = e^{i(\alpha+\varepsilon)}$. As in point (i), $d(F^k(x), F^k(y)) = 2^k\varepsilon$ so for k large enough, we will obtain $\pi > 2^k\varepsilon > \frac{\pi}{2}$ (find that k), so that the claim is proved.

Proof of (iii). The periodic points of F are any root of $F^k(z) - z$ for any $k > 0$. In particular, we solve $0 = F^k(z) - z = z^{2^k} - z = z(z^{2^k-1} - 1)$, so we find that the k-periodic points of F are the $2^k - 1$-th roots of unity, which we know form a regular $2^k - 1$-gon on the unit circle. As k increases, the number of vertices of this polygon becomes larger and larger and in the end, all the vertices of these polygons form a dense subset of the unit circle. Hence the proof. □

Additionally, it is instructive to look at the 2-cycles and 3-cycles of F.
The 2-periodic points of F on the unit circle are the third roots of unity $(1, e^{\frac{2i\pi}{3}}, e^{\frac{-2i\pi}{3}})$. Since $z = 1$ is a fixed point of its own, the remaining two roots for a 2-cycle $(e^{\frac{2i\pi}{3}}, e^{\frac{-2i\pi}{3}})$.
The 3-periodic points of F on the unit circle are seventh roots of unity $e^{\frac{2ik\pi}{7}}$ for $k = 0 \dots 6$. Again, $z = 1$ will not give a 3-cycle because we know it's a 1-cycle. The 6 remaining points will form two 3-cycles as shown in Figure 1.

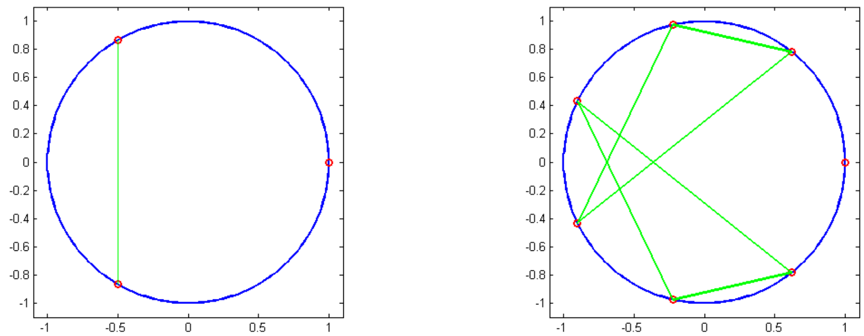

Figure 1: Two-cycles (left) and three-cycles (right) of the mapping $F(z) = z^2$.

Conjugacies and application to the study of Newton's algorithm for quadratic polynomials

A way to classify rational maps by means of their dynamical behavior is via the concept of *global conjugacy*:

Definition 2. *Given two rational maps $F, G : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and a linear fractional transformation ϕ, we say that F and G are* globally conjugate *via the conjugacy ϕ if $F \circ \phi = \phi \circ G$, i.e. the graph below commutes*

$$\begin{array}{ccc} \hat{\mathbb{C}} & \xrightarrow{G} & \hat{\mathbb{C}} \\ \phi \big\downarrow & & \big\downarrow \phi \\ \hat{\mathbb{C}} & \xrightarrow{F} & \hat{\mathbb{C}} \end{array}$$

You may check that being globally conjugate is an equivalence relation among rational maps. This implies the relation $G = \phi^{-1} \circ F \circ \phi$ or equivalently $F = \phi \circ G \circ \phi^{-1}$. Now let us state a few properties justifying why two globally conjugate maps exhibit the same dynamical behavior. First notice that if $G = \phi^{-1} \circ F \circ \phi$, then a simple calculation shows that

$$G^2 = \phi^{-1} \circ F \circ \underbrace{\phi \circ \phi^{-1}}_{Id} \circ F \circ \phi = \phi^{-1} \circ F^2 \circ \phi,$$

so that one can prove by induction that

$$G^k = \phi^{-1} \circ F^k \circ \phi, \qquad \forall k \in \mathbb{N}.$$

Theorem 2. *Let F, G be conjugate via the relation $G = \phi^{-1} \circ F \circ \phi$. Then the following properties hold true:*

(i) For any $k \geq 1$, $z^\star$ is a k-periodic point of G if and only if $\phi(z^\star)$ is a k-periodic point of F. Moreover, their multipliers are that the same, i.e. $(G^k)'(z^\star) = (F^k)'(\phi(z^\star))$ [2].

(ii) For $z^\star$ an attracting fixed point of G, then by virtue of (i), $\phi(z^\star)$ is an attracting fixed point of F, and then

$$\phi(A_G(z^\star)) = A_F(\phi(z^\star)).$$

(iii) The set where G is chaotic is mapped bijectively by ϕ onto the set where F is chaotic.

Proof. We only treat points (i) and (ii). First notice that if $G = \phi^{-1} \circ F \circ \phi$, then for any $k \geq 1$, $G^k = \phi^{-1} \circ F^k \circ \phi$, so that G^k is globally conjugate to F^k via the same conjugacy ϕ. Equivalently, we have $\phi \circ G^k = F^k \circ \phi$.

Proof of (i): Fix any $k \geq 1$. If $z^\star = G^k(z^\star)$, then

$$\phi(z^\star) = \phi \circ G^k(z^\star) = F^k \circ \phi(z^\star) = F^k(\phi(z^\star)),$$

hence $\phi(z^\star)$ is a k-periodic point of F, the converse is also true by symmetry of the conjugacy relation. Differentiating the relation $\phi \circ G^k(z) = F^k \circ \phi(z)$, we obtain

$$\phi'(G^k(z))(G^k)'(z) = (F^k)'(\phi(z))\phi'(z).$$

Upon evaluating this equality at $z^\star$, using that $z^\star = G^k(z^\star)$ and the fact that $\phi(z^\star) \neq 0$ because ϕ is conformal, we arrive at $(G^k)'(z^\star) = (F^k)'(\phi(z^\star))$, hence the multipliers are equal.

Proof of (ii): Let $z \in A_G(z^\star)$. This means that $\lim_{n\to\infty} G^n(z) = z^\star$. Applying ϕ, we obtain

$$\phi(z^\star) = \phi\left(\lim_{n\to\infty} G^n(z)\right) = \lim_{n\to\infty} \phi \circ G^n(z) = \lim_{n\to\infty} F^n(\phi(z)).$$

So if $z \in A_G(z^\star)$, then $\phi(z) = A_F(\phi(z^\star))$. The converse is also true by symmmetry of the conjugacy relation, so that the result holds. $\square$

Classification of Möbius transformations. In the language of conjugacies, the classification of LFTs previously established can be reformulated as follows:

- If F is a Möbius transformation with a single fixed point (i.e., the parabolic case), then there exists $b \in \mathbb{C}$ such that F is globally conjugate to the mapping $z \mapsto z + b$.
- If F is a Möbius transformation with two fixed points, then there exists $\mathfrak{m} \in \mathbb{C} - \{0\}$ such that F is globally conjugate to the mapping $z \mapsto \mathfrak{m} z$. In this case, F is *elliptic* when $|\mathfrak{m}| = 1$, *hyperbolic* when $\mathfrak{m} \in (0, +\infty)$, *loxodromic* otherwise.

Newton's algorithm for quadratic polynomials. Let us now see how we can use conjugacies in order to study the dynamics of the Newton algorithm applied to quadratic polynomials, restricting to the case of two distinct roots. Consider the function $f(z) = (z-\alpha)(z-\beta)$ with $\alpha \neq \beta$. The corresponding Newton map is $F(z) = z - \frac{f(z)}{f'(z)}$[3]. After simplification, we find $F(z) = \frac{z^2-\alpha\beta}{2z-(\alpha+\beta)}$. We will show the following.

[2] This implies that their types (attracting, superattracting, neutral, repelling) are the same.

[3] notice that this expression would not change if we had multiplied f by a constant, so there is no loss of generality in assuming this form for f

Proposition 3. *Let $f(z) = (z-\alpha)(z-\beta)$ a quadratic polynomial with two distinct roots $\alpha \neq \beta$. Then the Newton map of f defined by $F(z) = z - \frac{f(z)}{f'(z)}$ is globally conjugate to the mapping $z \mapsto z^2$.*

Proof. Define the LFT $\phi(z) = \frac{-\beta z+\alpha}{-z+1}$, then a direct calculation shows that

$$F \circ \phi(z) = \phi(z^2) = \phi \circ G(z), \qquad G(z) = z^2. \tag{1}$$

□

Remark 2. *The construction of $\phi^{-1}(z) = \frac{z-\alpha}{z-\beta}$ is motivated by the fact that if ϕ were such a conjugacy, by virtue of Theorem 2, ϕ^{-1} would map fixed points of F to fixed points of G and the multipliers of these fixed points would have to be the same. This is indeed satisfied, as*

- *$G(z) = z^2$ has fixed points $(0, 1, \infty)$ with multipliers $(0, 2, 0)$.*
- *F has fixed points (α, β, ∞) with multipliers $(0, 0, 2)$.*

Therefore, ϕ^{-1} must map (α, β, ∞) to $(0, \infty, 1)$, which is exactly what it does.

Combining Proposition 3 and Theorem 2 with the analysis of the map $G(z) = z^2$, we may draw the following conclusions: for $F(z) = \frac{z^2-\alpha\beta}{2z-(\alpha+\beta)}$ the Newton map of the quadratic polynomial $f(z) = (z-\alpha)(z-\beta)$, we have that

- $A_F(\alpha) = A_F(\phi(0)) = \phi(A_G(0)) = \phi(\mathbb{D})$.
- $A_F(\beta) = A_F(\phi(\infty)) = \phi(A_G(\infty)) = \phi(\hat{\mathbb{C}} \backslash \overline{\mathbb{D}})$.
- F is chaotic on $\phi(\partial\mathbb{D})$.

A closer look at the mapping $\phi(z) = \frac{-\beta z+\alpha}{-z+1}$ tells us that $\phi(\partial\mathbb{D})$ is the median line between α and β, call it $L \cup \{\infty\}$ (note that $\phi(1) = \infty$, and since ϕ is an LFT, it must map the circle $\partial\mathbb{D}$ to a circle in $\hat{\mathbb{C}}$ passing through ∞, that is to say, a straight line). The points on this line are precisely where F is chaotic, i.e. where the Newton algorithm fails to converge to a root of f. A heuristic interpretation of this would be: if a seed is equidistant to both roots α and β, an argument of "symmetry" tells us that there is *a priori* no reason for this seed to be attracted more by one root or the other[4].

Our conclusion on the Newton algorithm for a quadratic polynomial with two distinct roots is that, under the dynamical system $F(z) = z - \frac{f(z)}{f'(z)}$, any seed has its orbit converge to whatever root of f is is closer to. If it is equidistant to both roots, then the algorithm fails to converge and the dynamics is, in fact, chaotic.

[4] We will see that this heuristic argument completely falls through in the case of a cubic polynomial.

Invariant sets: Julia and Fatou

Newton's algorithm for cubic polynomials. We've seen above that for any quadratic polynomial with distinct roots, the dynamical behaviors are quite similar to one another as they are all, in fact, pairwise conjugate. The case of cubic polynomials, involving now the interplay between three basins of attraction instead of two, not only leads to a higher diversity of behavior, but also higher complexity of sets.

A first observation is that not all cubic polynomials are pairwise conjugate, however an elementary family to study is the following family $p_\rho(z) = z(z-1)(z-\rho)$, with roots $(0, 1, \rho)$ with $\rho \in U = \{\text{Im } (z) > 0\} \cap \overline{\mathbb{D}} \cap \overline{D_1(1)}$.

Theorem 4. *For any polynomial with three distinct roots $f(z) = (z-\alpha)(z-\beta)(z-\gamma)$, there exists a linear transformation $T(z) = az + b$ and $\rho \in U$ such that the Newton map F is globally conjugate to the Newton map of p_ρ via T.*

Because ρ is now a free parameter, that leaves a lot of room for different dynamical behaviors !

Now one may wonder, when picking a cubic polynomial, thereby creating three superattracting fixed points in the Newton dynamics, is it still true that a seed will converge to the roots that it is closest to ? In the case of the polynomial $f(z) = z^3 - 1$ should we draw the three medians in between each pair of roots of unity and determine the basins of attraction in this way ? The answer turns out to be much more complicated, as one may see on Figure 2 (also see Fig. 3 for another example of cubic). As a foretaste, the main constraint leading to such complicated behavior is the following result, of which we will provide a proof later.

Theorem 5. *If F is a rational function and z_1, z_2 are two attracting fixed points of F, then necessarily*

$$\partial A_f(z_1) = \partial A_F(z_2).$$

In the case of $f(z) = z^3 - 1$, we thus require that $\partial A_f(1) = \partial A_f(e^{\frac{2\pi i}{3}}) = \partial A_f(e^{\frac{-2\pi i}{3}})$!

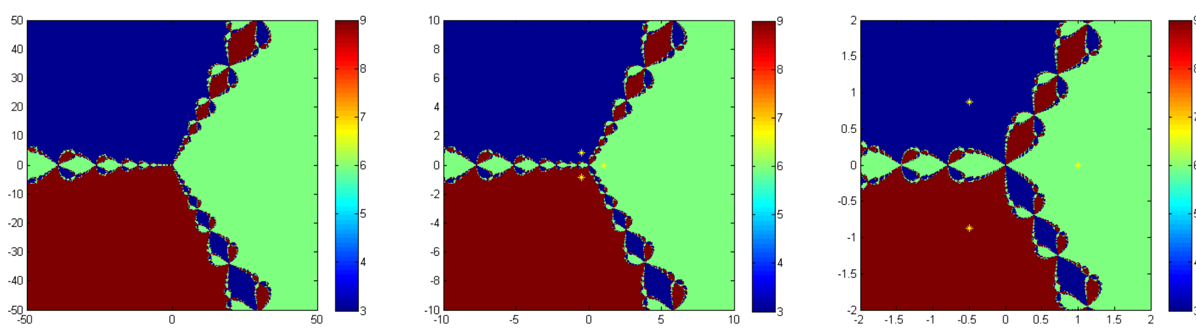

Figure 2: Partitioning of the plane into the basins of attractions of the roots of $z^3 - 1$. at different levels of zooming.

We will now introduce notions that will help us prove Theorem 5.

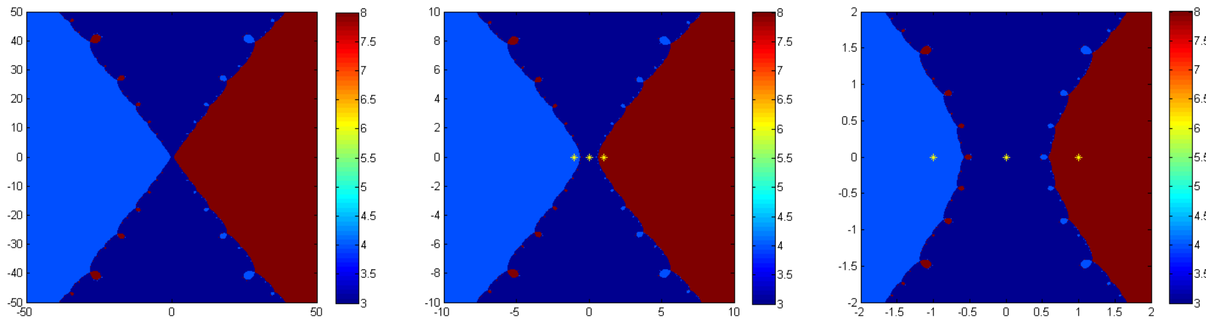

Figure 3: Partitioning of the plane into the basins of attractions of the roots of $z(z-1)(z+1)$, at different levels of zooming.

Topological preliminaries Here and below, we consider $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ analytic (i.e. a rational function) and U a subset of $\hat{\mathbb{C}}$. For $U \subset \hat{\mathbb{C}}$ we denote by $U^c = \hat{\mathbb{C}} - U$ the complement of U.

Definition 3. *We say that*

(i) U is invariant *(under F) if $F(U) \subseteq U$.*

(ii) U is completely invariant *(under F) if $F(U) \subseteq U$ and $F^{-1}(U) \subseteq U$.*

Example 1. *If $F(z) = z^2$, then $\mathbb{D}$ is completely invariant, and $\{1\}$ is invariant but not completely invariant, since $F(1) = 1$ but $F^{-1}(1) = \{-1, 1\}$.*

Lemma 6. *$F^{-1}(U) \subseteq U$ if and only if $F(U^c) \subseteq U^c$. As a consequence, U is completely invariant if and only if U^c is completely invariant.*

Proof. ($\Leftarrow$) Suppose $F(U^c) \subseteq U^c$. Let $y \in F^{-1}(x)$ where $x \in U$, in particular $f(y) = x$. If $y \in U^c$, then by the assumption, $x = f(y) \in U^c$ which is a contraction, therefore $y \in U$. Hence $F^{-1}(U) \subseteq U$.
($\Rightarrow$) Suppose $F^{-1}(U) \subseteq U$. Let $x \in U^c$. If $F(x) \in U$, we have $x \in F^{-1}(F(x)) \subseteq F^{-1}(U) \subseteq U$ which is a contraction, hence $F(x) \in U^c$. Hence $F(U^c) \subseteq U^c$. $\square$

Lemma 7. *If F is an open mapping (i.e. it maps open sets to open sets) and U is invariant, then U^o, the interior of U, is invariant.*

Proof. If $y \in U^o$ there exists an open set V such that $y \in V \subset U^o$. Then $F(V)$ is an open neighborhood of $F(y)$ included in U, that is, $F(y) \in U^o$. $\square$

Lemma 8. *If F is a continuous open mapping*[5]*, and U is a completely invariant open set, then so is ∂U.*

Proof. Since U is open then $U \cap \partial U = \emptyset$. Let $x \in \partial U$. There exists $x_n \in U$ such that $x_n \to x$. Since $F(U) \subseteq U$, the sequence $F(x_n)$ is in U and therefore its limit is in $\overline{U} = U \cup \partial U$. Since U is completely invariant and $x \notin U$, $F(x) \notin U$ so $F(x) \in \partial U$. Therefore $F(\partial U) \subseteq \partial U$.
Let us now prove that $F^{-1}(\partial U) \subseteq \partial U$ by showing that $F((\partial U)^c) \subseteq (\partial U)^c$. Indeed, $(\partial U)^c$ is the disjoint union of U (which is F-invariant) and $(U^c)^o$, which is F-invariant since U^c itself is and using the previous lemma. $\square$

[5]The Open Mapping Theorem says that this is always satisfied when F is analytic and non-constant.

We finish the preliminaries by stating without proof another theorem of Montel's theorem, establishing the property of a sequence of meromorphic functions to have a normally convergent subsequence, under much weaker assumptions.

Theorem 9 (Montel's theorem (hard version))**.** *A sequence of meromorphic functions $f_n : U \to \hat{\mathbb{C}}$ which omits three values in $\hat{\mathbb{C}}$ (in the sense that the set $\hat{\mathbb{C}} - \bigcup_{n\in\mathbb{N}} f_n(U)$ contains at least 3 distinct points) is normal.*

Remark 3. *Note that if a sequence is uniformly bounded (the condition for the first Montel's theorem), then it clearly omits at least three points (in fact $\hat{\mathbb{C}} - \bigcup_{n\in\mathbb{N}} f_n(U)$ is an unbounded set !).*

Julia and Fatou sets. We are now ready to introduce the concept of Fatou and Julia sets, named after Gaston Julia and Pierre Fatou, who initiated the study of iterated rational functions in the 1920's.

Definition 4 (Julia and Fatou sets)**.** *Given $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a rational function, we define the Fatou set of F (denoted $\mathcal{F}(F)$) as: $z_0 \in \mathcal{F}(F)$ if there exists $\rho > 0$ such that the sequence of iterates $\{F^n\}_n$ is normal on $D_\rho(z_0)$. The Julia set of F (denoted $\mathcal{J}(F)$) is the complement in $\hat{\mathbb{C}}$ of $\mathcal{F}(F)$.*

We will drop the F when notation is clear from the context. By construction, the Fatou set is open, then the Julia set is closed. The Fatou set is where the dynamics is "well-behaved"/tame, whereas the Julia set is where the dynamics is chaotic.

Proposition 10. *The Fatou set $\mathcal{F}(F)$ is completely invariant under F. As a consequence, so is the Julia set $\mathcal{J}(F)$.*

Proof. We must show for every $z_0 \in \hat{\mathbb{C}}$, $z_0 \in \mathcal{F}(F)$ if and only if $f(z_0) \in \mathcal{F}(F)$.
($\Longleftarrow$) Suppose $f(z_0) \in \mathcal{F}(F)$, then there exists U an open neighborhood of $f(z_0)$ and a sequence $\{n_k\}_k$ of integers where the sequence $\{F^{n_k}\}_k$ converges normally. Then $f^{-1}(U)$ is an open neighborhood of z_0 where the sequence $\{F^{n_k+1}\}_k$ converges normally, hence $z_0 \in \mathcal{F}(F)$.
($\Longrightarrow$) Similar proof, except that one must use the fact that F is an open mapping to say that it maps an open neighborhood of z_0 into an open neighborhood of $f(z_0)$. □

Some properties of the Julia set. For an integer n, we denote the set

$$F^{-n}(w) := F^{-1}(F^{-1} \dots (F^{-1}(w))),$$

where the preimage is taken n times.

Proposition 11. *For $w \in \mathcal{J}$, the set $\bigcup_{n\in\mathbb{N}} F^{-n}(w)$ is dense in $\mathcal{J}$.*

Proof. Let $w \in \mathcal{J}$. We want to show that for every $z \in \mathcal{J}$ and every neighborhood $V \ni z$, there exists k such that $F^{-k}(w) \cap V \neq \emptyset$. By the contraposition of Montel's theorem above, $\{F^n\}_n$ is not normal on V so $\mathbb{C} - \bigcup_n F^n(V)$ is at most two points and it can be shown by exhaustion of cases that these exceptional points belong to the Fatou set, so w is not one of them. Therefore, $w \in \bigcup_n F^n(V)$, i.e. there exists k such that $w \in F^k(V)$. That means that $F^{-k}(w) \cap V \neq \emptyset$, hence the result. □

Example 2. *For $F(z) = z^2$, $\mathcal{J}(F) = \partial\mathbb{D}$, then we have seen in a previous exercise that $\bigcup_{n\in\mathbb{N}} F^{-n}(1)$ is the union over $n \in \mathbb{N}$ of all 2^n-th roots of 1. This set is clearly dense in the unit circle.*

Using Proposition 11, we can establish the following

Proposition 12. *$\mathcal{J}$ has no proper subset that is both closed and completely invariant.*

Proof. Let K be a subset of $\mathcal{J}$ satisfying both properties, and let $w \in K$. Then since K is completely invariant, $\bigcup_{n\in\mathbb{N}} F^{-n}(w) \subset K$. Taking topological closure and using the previous proposition, we obtain that

$$\mathcal{J} \subseteq \overline{\bigcup_{n\in\mathbb{N}} F^{-n}(w)} \subseteq \overline{K} = K,$$

hence $K = \mathcal{J}$. □

Application to Newton's algorithm for cubic polynomials We can now return to the proof of theorem 5. As mentioned in lecture 22, if $f(z) = (z - w_1)(z - w_2)(z - w_3)$ is a cubic polynomial and F is its Newton map, F has three superattracting fixed points at w_1, w_2, w_3, each of which has an open basin of attraction $A_F(w_k)$. Each basin of attraction is completely invariant, so by the preliminaries, $\partial A_F(w_k)$ is completely invariant as well. One can also show that this set is closed, and a quick moment's thought makes one think that it could not be part of the Fatou set (otherwise, that basin of attraction could be extended a little further). Therefore for each $k = 1, 2, 3$, $\partial A_F(w_k)$ is a closed, completely invariant subset of $\mathcal{J}$, which by virtue of Proposition 12, implies that

$$\partial A_F(w_1) = \partial A_F(w_2) = \partial A_F(w_3) = \mathcal{J}(F).$$

Enforcing this condition clearly leads to the complicated intertwining that is occuring on the pictures (see Figure 4) showing basins of attraction: any neighborhood of any point in $\mathcal{J}(F)$ must contains points which belong to each of the basins of attraction !

To generalize further, Newton's dynamics for polynomials of degree $d \geq 3$ may be such that the Julia set must be the common boundary to d disjoint basins of attractions.

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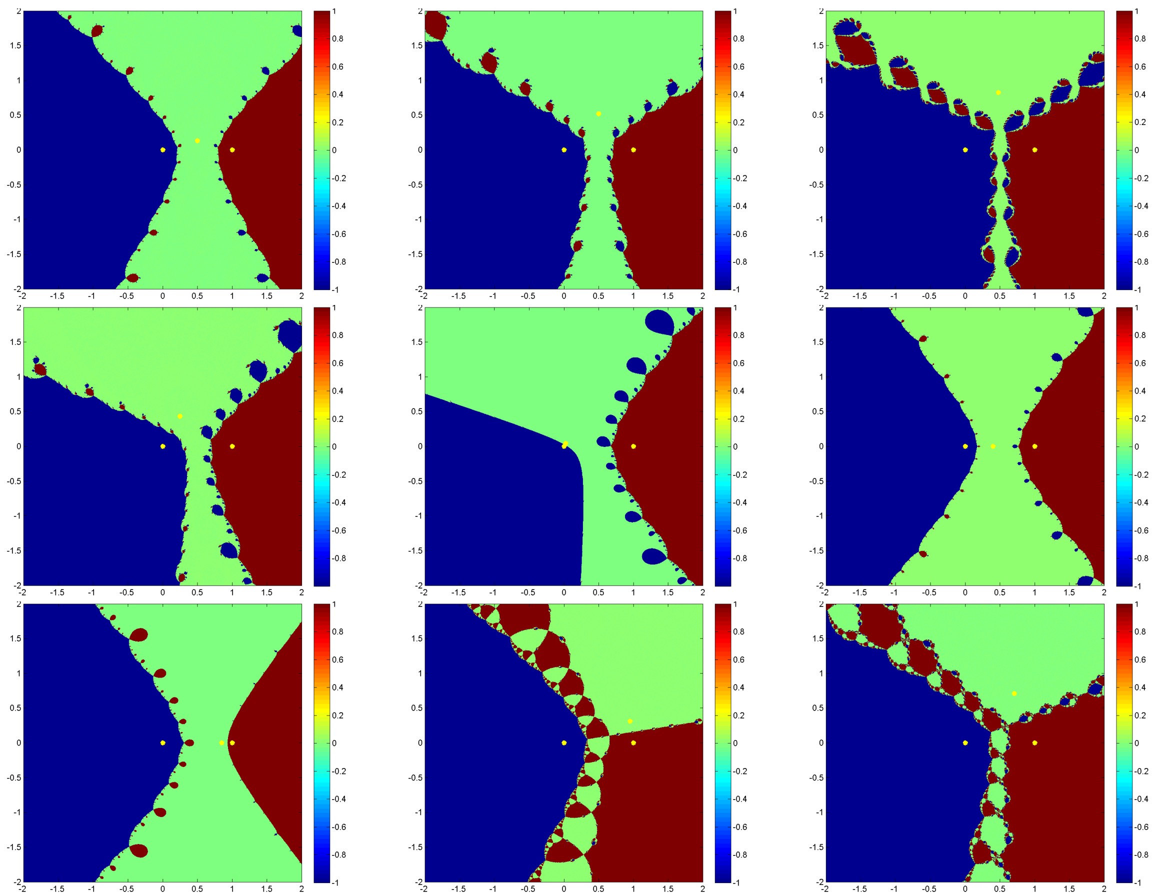

Figure 4: More examples of basins of attractions for the Newton dynamics of various sample cubic polynomials. The roots are in yellow.