MATH 207 - Spring 2017 - HW4 - due 04/27

Turn in:

1. If $f(z) = \frac{z^4}{z^2-1}$ is considered as an analytic function from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$, where are the poles of f and what are their orders ?

2. Find the critical points and critical values of $f(z) = z + \frac{1}{z}$. Sketch the curves where $f(z)$ is real. Sketch the regions where Im $f(z) > 0$ and where Im $f(z) < 0$.

3. Suppose that $f : \Omega \to \mathbb{C}$ is an analytic function such that there exists z_0 and $r > 0$ such that $f(z) \in \mathbb{R}$ for every $z \in D_r(z_0)$. Prove that f is constant. [Hint: Open Mapping theorem]

4. Let $f(z)$ a polynomial of degree m.
   (a) How many critical points, counting multiplicity, does f have in $\mathbb{C}$? Justify your answer.
   (b) Find and plot the critical points and critical values of $g(z) = z^2 + 1$ and of its iterates $g(g(z)) = (z^2+1)^2 + 1$ and $g(g(g(z)))$. [Hint: use the chain rule]
   (c) Let $f(z)$ a polynomial of degree m. How many (finite) critical points does the N-fold iterate $f \circ \cdots \circ f$ (N times) have ? Describe them in terms of the critical points of f.

5. (a) Prove that the only LFT mapping the triple $(0, 1, \infty)$ into itself is the mapping $h(z) = z$.
   (b) Given $w_1, w_2, w_3 \in \mathbb{C}$ distinct, find an LFT mapping (w_1, w_2, w_3) to $(0, 1, \infty)$.
   (c) Using (a) and (b) prove that given two triples (z_1, z_2, z_3) distinct and (w_1, w_2, w_3) distinct, there exists a unique LFT h such that $h(z_j) = w_j$ for $j = 1, 2, 3$.
   (d) Compute explicitly the LFT's determined by the following correspondence of triples:
      i. $(1+i, 2, 0) \mapsto (0, \infty, i-1)$
      ii. $(1, i, -1) \mapsto (1, 0, -1)$

For your practice:

1. Show that the function $f(z) = \frac{1}{z^2+5}$ has a limit at ∞, find it, and show that f is analytic at ∞.

2. Can we consider $\sin z$ as an analytic function from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$? Justify your answer.

3. Prove that for a rational function, there exists a fixed integer d such that every value in $\hat{\mathbb{C}}$ is achieved exactly d times.

4. Find an LFT that takes the line $\{\Re z = 1\}$ to the circle of radius one centered at -1.

5. What is the image of the unit disk under the LFT $h(z) = \frac{2iz}{z-1}$?

6. What is the image of the disk of radius 1, centered at $1+i$, under the LFT $h(z) = \frac{1+z}{1-z}$?