Math 216 partial recap - 01/27/216 - François Monard

This summary covers some of the topics we have been covering in class, in a somewhat compact form. These notes are BY NO MEANS reflective of what may or may not appear in the midterm and should not be considered as a replacement for the book or lecture notes. However, as today was a "review" lecture, this might be a good time to provide some brief checkpoint notes.

Methods for solving ODEs

The general setting is an initial value problem of the form:

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0.$$

In certain cases[1], we are able to work out an explicit expression for the solution, depending on the form of f.

Case 1: f does not depend on y. Then we are looking at a problem of the form $\frac{dy}{dx} = f(x)$, in which case finding y consists of finding an antiderivative for f.

Case 2: f is separated, that is, $f(x,y) = g(x)h(y)$. In this case, we may rewrite the ODE as

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x).$$

The task now is to find an antiderivative for $\frac{1}{h}$ (call it H) and an antiderivative for g (call it G), so that the previous equation reads:

$$\frac{d}{dx}(H(y(x))) = \frac{dH}{dy}\frac{dy}{dx} = \frac{1}{h(y)}\frac{dy}{dx} = g(x) = \frac{d}{dx}G(x).$$

Then, direct integration gives $H(y(x)) = G(x) + C$, after which one might be able to solve for $y(x)$, and the integration constant C using the initial condition.

Case 3: f is linear in y, of the form $f(x,y) = -P(x)y + Q(x)$. In this case, the ODE takes the form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

The recipe to solve the problem goes as follows:

[1] Note that in real life, most cases cannot be solved explicitely !

1. Compute an *integrating factor* $\rho(x) = \exp\left(\int P(x)\ dx\right)$, which consists of finding an antiderivative for P, then taking its exponential.

2. Notice, using the product rule, that the ODE can be turned into
$$\frac{d}{dx}(\rho(x)y(x)) = \rho(x)Q(x).$$

3. Integrate this equation once. This will create an integration constant.

4. solve for y and for the integration constant using the initial condition.

Tricks for finding antiderivatives. In all examples above, the main technical step is to find antiderivatives. While many of them are easy or can be found in tables, some others may not be so obvious. One trick to be known, is to recognize when an expression takes the form of the derivative of a compound, for instance:

- If an expression looks like $2f(t)f'(t)$ for some function f, then it is nothing but $\frac{d}{dt}(f^2(t))$, in which case an antiderivative is $f^2(t)$.
- If an expression looks like $f'(t)/f(t)$ for some function f, then it is nothing but $\frac{d}{dt}(\ln f(t))$, in which case an antiderivative is $\ln(f(t))$.
- If an expression takes the form $-f'(t)/(f(t))^2$ for some function f, then it is nothing but $\frac{d}{dt}(1/f(t))$.

In all these cases, we recognized that the expression took the form $(g'(f(t))f'(t) = \frac{d}{dt}(g(f(t)))$ (with $g(x)$ respectively x^2, $\ln x$, $1/x$), for which an antiderivative is given by $g(f(t))$. To practice recognizing such forms, you want to try other cases such as $g(x) = x^p$ for any power...

As examples, try looking for antiderivatives of $\frac{\cos t}{\sin t}$, $\frac{1}{t \ln t}$, $\frac{\ln t}{t}$, $\frac{3(\ln t)^2}{t}$, $3t^2 \sin(t^3)$, using the methods above.

Parameter-dependent ODEs and bifurcations

Autonomous ODEs, equilibrium solutions and stability. An ODE $\frac{dy}{dx} = f(x,y)$ is called *autonomous* if f does not depend on the independent variable x, that is, the ODE looks like $\frac{dy}{dx} = f(y)$. In what follows, we will restrict to ODEs of this form. For such an ODE, we define $y^\star$ to be a *critical point* if $f(y^\star) = 0$. For any critical point, the constant function $y(x) = y^\star$ is a solution curve to the problem, so-called an "equilibrium curve", hence their importance. Intuitively[2], a critical point $y^\star$ is called

[2]See the book for more precise statements.

- *stable* if nearby solution curves seem to remain close to $y^\star$.
- *semi-stable* if nearby solution curves converge to that point on one side of it, diverge away on the other side (example: the critical point $y^\star = 0$ in the ODE $dy/dx = y^2$).
- *unstable* otherwise.

As you may notice by plotting examples of velocity fields,

- $y^\star$ is unstable if $f(y) > 0$ for y near and above $y^\star$, and $f(y) < 0$ for y near and below $y^\star$.
- $y^\star$ is stable if $f(y) > 0$ for y near and below $y^\star$, and $f(y) < 0$ for y near and above $y^\star$.
- $y^\star$ is semi-stable if f keeps the same sign over an interval containing $y^\star$.

Bifurcations. Some models may be made *parameter-dependent*, in the sense that the ODE may involve additional variables which are part of the model. We covered for instance the case of a logistic population with harvesting

$$\frac{dy}{dx} = y(4-y) - \lambda,$$

where the parameter $\lambda \geq 0$ quantifies a constant harvesting rate of the population y. Despite the fact that everything in the model looks continuous in term of λ, one may behold drastic (or discontinuous) changes in the number of critical points and/or in the stability properties of these critical points, as λ passes through certain values. We call such phenomena *bifurcations*. Bifurcations are best observed on a bifurcation diagram, obtained as follows: given a parameter-dependent, autonomous ODE, of the form $\frac{dy}{dx} = f_\lambda(y)$,

1. For each value of λ, find the critical points $y^\star(\lambda)$ by solving for y the equation $f_\lambda(y) = 0$, and determine the sign of f_λ in-between the critical points to decide which way trajectories go. From this you are able to decide the type of each critical point (i.e., stable, semi-stable or unstable).
2. Draw the functions $y^\star(\lambda)$ on a (λ, y)-plane. This is the bifurcation diagram.

Summary of numerical methods

We have seen three kinds of numerical schemes in order to approximate a solution of an initial value problem of the form

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0.$$

We suppose that there exists a unique solution $y(x)$ to the problem above, defined on some interval containing x_0.

The three methods start from a stepsize $h > 0$, and aim at constructing a sequence of approximations $y_0, y_1, y_2, \dots$ of $y(x_0), y(x_1), y(x_2)$ where we define $x_n := x_0 + nh$ for a finite number of steps $0 \le n \le N$. We say that a method is accurate at order p (an integer) if there exists a constant C such that

$$|y(x_n) - y_n| \le Ch^p, \qquad 0 \le n \le N.$$

The higher the p, the faster the approximations converge to the true solution as one refines h.

These methods go as follow.

Euler: (first-order accurate)

$$y_{n+1} = y_n + hf(x_n, y_n), \qquad n = 0, 1, 2, \dots$$

Improved Euler: (second-order accurate)

$$\begin{aligned} k_1 &= f(x_n, y_n), \\ k_2 &= f(x_{n+1}, y_n + k_1 h), \\ y_{n+1} &= y_n + h\frac{k_1 + k_2}{2}, \qquad n = 0, 1, 2, \dots \end{aligned}$$

Runge-Kutta 4: (fourth-order accurate)

$$\begin{aligned} k_1 &= f(x_n, y_n), \\ k_2 &= f(x_n + h/2, y_n + (h/2)k_1), \\ k_3 &= f(x_n + h/2, y_n + (h/2)k_2), \\ k_4 &= f(x_{n+1}, y_n + hk_3), \\ y_{n+1} &= y_n + h\frac{k_1 + 2k_2 + 2k_3 + k_4}{6}, \qquad n = 0, 1, 2, \dots \end{aligned}$$

As seen in class, these methods are motivated, in the abstract, by more and more accurate rules for approximating a given function over a small interval: indeed, integrating the ODE between x_n and x_{n+1}, we see that the true solution satisfies

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(t, y(t))\ dt,$$

and, while we cannot compute the rightmost term exactly (we don't know $y(t)$ for $x_n \le t \le x_{n+1}$!), all the schemes above are based on increasingly accurate rules for computing this integral (respectively, the rectangle rule, the trapezoidal rule, and Simpson's rule).