Math 216, lecture 33
Dynamical systems, chaos and fractals

François Monard

Department of Mathematics, University of Michigan

03/30/2016

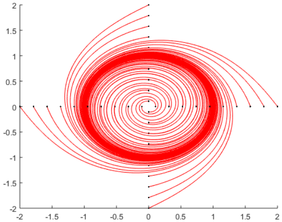
Under general assumptions of F, G, the long-term outcome of a trajectory of $(x(t), y(t))$ falls into either of the four cases:

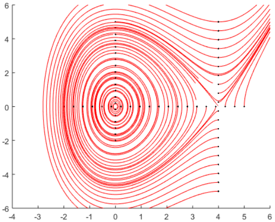
1. $(x(t), y(t))$ converges to a critical point as $t \to \infty$.
2. $(x(t), y(t))$ goes unbounded.
3. $(x(t), y(t))$ is a periodic solution with a closed trajectory.
4. $(x(t), y(t))$ spirals toward a closed trajectory.

This means that two-dimensional trajectory are rather "predictable".
(no "chaos" in 2D autonomous continuous-time system)

Under general assumptions of F, G, the long-term outcome of a trajectory of $(x(t), y(t))$ falls into either of the four cases:

1. $(x(t), y(t))$ converges to a critical point as $t \to \infty$.
2. $(x(t), y(t))$ goes unbounded.
3. $(x(t), y(t))$ is a periodic solution with a closed trajectory.
4. $(x(t), y(t))$ spirals toward a closed trajectory.



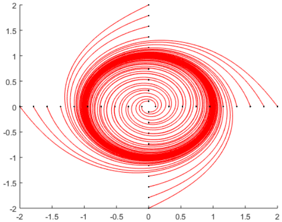


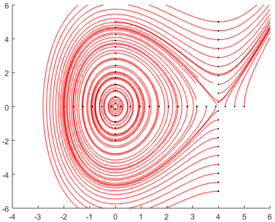
This means that two-dimensional trajectory are rather "predictable".

(no "chaos" in 2D autonomous continuous-time system)

Under general assumptions of F, G, the long-term outcome of a trajectory of $(x(t), y(t))$ falls into either of the four cases:

1. $(x(t), y(t))$ converges to a critical point as $t \to \infty$.
2. $(x(t), y(t))$ goes unbounded.
3. $(x(t), y(t))$ is a periodic solution with a closed trajectory.
4. $(x(t), y(t))$ spirals toward a closed trajectory.



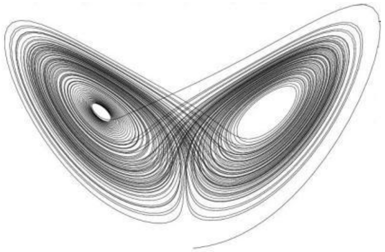


This means that two-dimensional trajectory are rather "predictable".
(no "chaos" in 2D autonomous continuous-time system)

In higher dimensions, "predictability" of long-term behavior of trajectories does not always hold.

The Lorenz system:

$$x' = \sigma(y - x)$$
$$y' = \rho x - y - xz$$
$$z' = -\beta z + xy$$



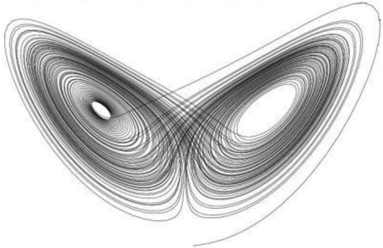
The Lorenz "strange attractor"

One cannot predict when the trajectory switches attracting component.

In higher dimensions, “predictability” of long-term behavior of trajectories does not always hold.

The Lorenz system:

$$x' = \sigma(y - x)$$
$$y' = \rho x - y - xz$$
$$z' = -\beta z + xy$$



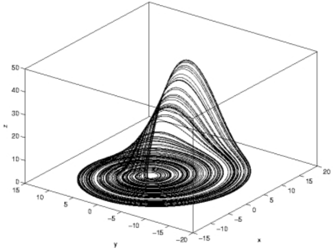
The Lorenz “strange attractor”

One cannot predict when the trajectory switches attracting component.

In higher dimensions, "predictability" of long-term behavior of trajectories does not always hold.

Rössler system:

$$x' = -y - z$$
$$y' = x + \alpha y$$
$$z' = \beta - \gamma z + xz$$



The Rössler band

Back to 2D, can we achieve chaos/unpredictability in any way ?

Yes, by moving from **continuous**-time dynamical systems to **discrete**-time dynamical systems.
Example:

$$
\begin{aligned}
x_{n+1} &= f(x_n, y_n), \\
y_{n+1} &= g(x_n, y_n).
\end{aligned}
$$

For such systems:

- One can define again critical points and their stability (not by solving $f = g = 0$!),
- The plane is partitioned between basins of attraction of stable critical points
- At the interface lies unpredictability.

Back to 2D, can we achieve chaos/unpredictability in any way ?

Yes, by moving from **continuous**-time dynamical systems to **discrete**-time dynamical systems.

Example:

$$
\begin{aligned}
x_{n+1} &= f(x_n, y_n), \\
y_{n+1} &= g(x_n, y_n).
\end{aligned}
$$

For such systems:

- One can define again critical points and their stability (not by solving $f = g = 0$!),
- The plane is partitioned between basins of attraction of several critical points.
- At the interface lies unpredictability.

Back to 2D, can we achieve chaos/unpredictability in any way ?

Yes, by moving from **continuous**-time dynamical systems to **discrete**-time dynamical systems.
Example:

$$x_{n+1} = f(x_n, y_n),$$
$$y_{n+1} = g(x_n, y_n).$$

For such systems:

- One can define again critical points and their stability (not by solving $f = g = 0$!),
- The plane is partitioned between basins of attraction of stable critical points.
- At the interface lies unpredictability.

Back to 2D, can we achieve chaos/unpredictability in any way ?

Yes, by moving from **continuous**-time dynamical systems to **discrete**-time dynamical systems.
Example:

$$x_{n+1} = f(x_n, y_n),$$
$$y_{n+1} = g(x_n, y_n).$$

For such systems:

- One can define again critical points and their stability (not by solving $f = g = 0$!),
- The plane is partitioned between basins of attraction of stable critical points.
- At the interface lies unpredictability.

Back to 2D, can we achieve chaos/unpredictability in any way ?

Yes, by moving from **continuous**-time dynamical systems to **discrete**-time dynamical systems.
Example:

$$x_{n+1} = f(x_n, y_n),$$
$$y_{n+1} = g(x_n, y_n).$$

For such systems:

- One can define again critical points and their stability (not by solving $f = g = 0$!),
- The plane is partitioned between basins of attraction of stable critical points.
- At the interface lies unpredictability.

Back to 2D, can we achieve chaos/unpredictability in any way ?

Yes, by moving from **continuous**-time dynamical systems to **discrete**-time dynamical systems.
Example:

$$
\begin{aligned}
x_{n+1} &= f(x_n, y_n), \\
y_{n+1} &= g(x_n, y_n).
\end{aligned}
$$

For such systems:

- One can define again critical points and their stability (not by solving $f = g = 0$!),
- The plane is partitioned between basins of attraction of stable critical points.
- At the interface lies unpredictability.

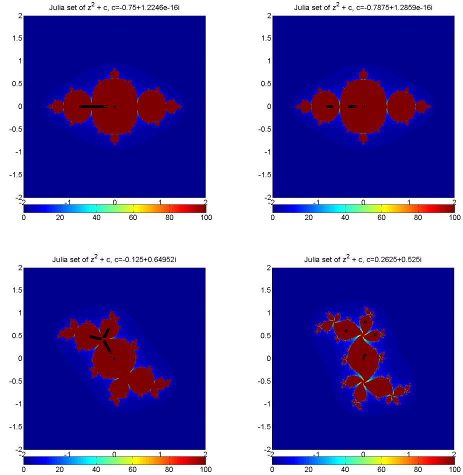
An example, the quadratic family:

$$x_{n+1} = 2x_n y_n + a,$$
$$y_{n+1} = x_n^2 - y_n^2 + b.$$

For some values of (a, b), the plane is divided into two regions:

- one where the trajectories accumulate at a point, another where the trajectories accumulate at ∞.
- At the interface, the dynamics is so-called "chaotic" (unpredictable, no "trend" or collective behavior of trajectories).

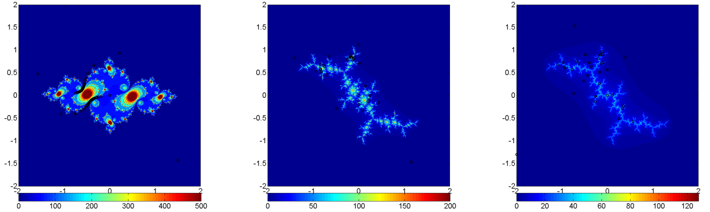
Examples of partitionings for four such values of (a, b).



The interface is where the unpredictable (or chaotic) trajectories occur. This forces the interface to be very "rough" ($\rightarrow$ fractals).

For some other values of (a, b), the plane is divided between two regions:

- A single region where trajectories collectively accumulate at ∞.
- A "dust" of unpredictable orbits.



This dichotomy of cases splits the (a, b)-plane into two regions, one of them is called the **Mandelbrot** set, a set under active study.