4.1 Composite Functions (Cont'd)

\[ f \circ g (x) = f(g(x)) \]

Domain of \( f \circ g \) is set of all numbers \( x \) is in the domain of \( g \) and \( g(x) \) is in domain of \( f \).

Example:

\[ f(x) = \sqrt{x - 4}, \quad g(x) = \frac{6}{x - 3} \]

\[ f \circ g (4) = f(g(4)) = f(\frac{6}{4-3}) = f(2) = \sqrt{2} - 4 = \sqrt{2} \]

\[ f \circ g (2) = f(g(2)) = f(\frac{6}{2-3}) = f(-6) = \sqrt{-6} - 4 \]

\[ = \sqrt{-10} \quad \text{(No Real Solution)} \]

**Find**

Domain of \( f \circ g \) = Domain of \( g \) = \( \{ x \mid x \neq 3 \} \)

\( g(x) \) is in domain of \( f \): \( x - 4 \geq 0 \)

\[ x \geq 4 \]
So, \( g(x) = \frac{6}{x-3} \geq 4 \)

\[
\frac{6}{x-3} - 4 \frac{(x-3)}{1(x-3)} \geq 0
\]

\[
\frac{6 - 4(x-3)}{x-3} \geq 0
\]

\[
\frac{6 - 4x + 12}{x-3} \geq 0
\]

\[
\frac{2 - 2}{x-3} - 4x + 18 \geq 0
\]

\[
\frac{-4x + 18}{x-3} \geq 0
\]

\[
-4x + 18 = 0
\]

\[
\frac{-4x}{4} + \frac{18}{4} = x = \frac{9}{2}
\]

\[
x - 3 = 0 \Rightarrow x = 3
\]

Domain of \( fog \):

\[
(3, \frac{9}{2}]
\]

Ex. For \( fog \):

\[
f(x) = \sqrt{x-4} \quad g(x) = \frac{6}{x-3}
\]

Find Domain of \( gof \)
Domain of \( g \circ f \): \( x \in \text{Domain of } f \)

\( g \circ f(x) = g(f(x)) \) and \( f(x) \in \text{Domain of } g \)

Domain of \( f = \{ x \mid x \geq 4 \} = [4, \infty) \)

\( x - 4 \geq 0 \)

\( x \geq 4 \)

Need \( f(x) \) in domain of \( g \)

\( f(x) \neq 3 \)

\( \sqrt{x - 4} \neq 3 \)

Squaring:

\( x - 4 = 9 \)

\( x = 13 \)

Domain of \( g \circ f = [4, 13) \cup (13, \infty) \)

\( g \circ f(8) = g(f(8)) = g(2) = \frac{6}{\sqrt{2} - 3} = \frac{6}{-1} = -6 \)
\[ f(x) = \sqrt{x - 4} \Rightarrow f(\ ) = \sqrt{(\ ) - 4} \]

\[ f \circ f(20) = f(f(20)) = f(4) = \sqrt{4 - 4} = 0 \]

\[ f \circ f(x) = f(f(x)) = f(\sqrt{x - 4}) = \sqrt{(\sqrt{x - 4} - 4} \]

**Note:** in general, \( f \circ g \neq g \circ f \)

\[ f \circ g(x) = f(g(x)) \text{ vs } g \circ f(x) = g(f(x)) \]

Sometimes \( f \circ g = g \circ f \).

\[ f(x) = 2x - 8 \quad g(x) = \frac{1}{2}x + 4 \]

\[ f \circ g(x) = f(g(x)) = f\left(\frac{1}{2}x + 4\right) = 2\left(\frac{1}{2}x + 4\right) - 8 = x + 8 - 8 = x \]

\[ g \circ f(x) = g(f(x)) = g(2x - 8) = \frac{1}{2}(2x - 8) + 4 = x - 4 + 4 = x \]

SAME.
4.2 ONE-TO-ONE FUNCTIONS: INVERSE FUNCS.

**Def:** A function $f$ is one-to-one if any two different inputs correspond to different outputs.

I.e., if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

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\{ ONE-TO-ONE \}

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\{ NOT ONE-TO-ONE \}

$x_1 = 2, x_2 = 4$

$2 \neq 4$ but $f(2) = f(4)$

$s = 5$

**Think (Horizontal Line Test):** If every horizontal line crosses the graph of $f(x)$ at most once, then $f$ is one-to-one.

\[
\begin{align*}
  y &= mx + b \\
  f(x) &= mx + b
\end{align*}
\]

\( (0, b) \)

ONE-TO-ONE
\[ y = x^2 \]

NOT ONE-TO-ONE.

\[ f(x) = x^2 \]
\[ f(s) = (s)^2 = 2s \]

AND
\[ f(-s) = (-s)^2 = 2s \]

NOTE: \( f(x) = x^2 \)  \( x \leq 0 \)

THIS IF \( f \) IS INCREASING ON \( I \) THEN \( f \) IS ONE-TO-ONE ON \( I \)

SAME: IF \( f \) IS DECREASING ON \( I \) THEN \( f \) IS 1 TO 1
**Definition**: Suppose $f$ is one-to-one. Then, corresponding to each $x \in \text{domain of } f$ there is exactly one $y$ in range. So corresponding to each $y$ in range of $f$ is exactly one $x$. This correspondence from range of $f$ to domain of $f$ is a function. It is called the inverse function of $f$, denoted $f^{-1}(x)$.

**Table 1**

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$\text{Domain of } f^{-1} = \text{Range of } f$ / $\text{Range of } f^{-1} = \text{Domain of } f$
Recall: \( f(x) = 2x - 8 \)  \( g(x) = \frac{1}{2}x + 4 \)

\( f(5) = 2(5) - 8 = 2 \)  \( g(2) = \frac{1}{2}(2) + 4 = 5 \)

\( g \circ f(x) = x \) (Last Section)

\( \text{And } f \circ g(x) = x \)  \( \text{Claim } g^{-1}(x) = f^{-1}(x) \)

\( \text{Note! } f \circ f^{-1}(x) = x \) for all \( x \in \text{Domain of } f \)

\( \text{And } f \circ f^{-1}(x) = x \) for all \( x \in \text{Domain of } f \)

Process to find \( f^{-1}(x) \):

1. \( y = f(x) \)
2. Interchange values for \( x \) and \( y \) \( x \leftrightarrow y \)
3. Solve new equation for \( y \) \( \text{[New formula for } y \text{ in terms of } x \text{] } \)
4. This formula is \( f^{-1}(x) \)

\( f(x) = 2x - 8 \)  \( y = 2x - 8 \)

\( x + 8 = 2y \)

\( \frac{x + 8}{2} = y \)

Switch: \( x = 2(y) - 8 \)  \( f^{-1}(x) = \frac{1}{2}x + 4 = y \)
Example: If \( g(x) = \frac{x+3}{2x-5} \), find \( g^{-1}(x) \)

\[
y = \frac{x+3}{2x-5}
\]

1) Switch \( x \leftrightarrow y \)

\[
(x) = \frac{(y)+3}{2(y)-5}
\]

2) Solve for \( y \)

\[
(2y-5)x = y + 3
\]

\[
2yx - 5x = y + 3
\]

\[
y = \frac{5x+3}{2x-1}
\]

\( f(x) = \frac{5x+3}{2x-1} \)

Example: \( y = x^2 \) \( x \leq 0 \)

\[
x = y^2 \rightarrow \sqrt[2]{x} = y
\]

Two choices.