3.2 ZEROS OF POLYNOMIALS (CONT'D)

DIVISION ALGORITHM: Given \( f(x), g(x) \)

Degree of \( g(x) > 0 \) there are unique polynomials \( q(x), r(x) \) such that

\[
\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{s(x)} \quad ; \quad \text{deg } r < \text{deg } g
\]

Alternate notation:

\[
f(x) = q(x) \cdot g(x) + r(x)
\]

Note if \( g(x) = x-c \) then \( r(x) = \text{constant} \) (degree 0)

REMAINDER THEOREM: If \( f(x) \) is divided by \( x-c \)

Remainder is \( f(c) \), constant.

Substituting:\

\[
f(x) = q(x)(x-c) + r
\]

\[x = c\]

\[
f(c) = q(c)((x-c)^0) + r = r \quad \checkmark
\]
THM: If a polynomial \( f(x) \) has a root at \( x = c \), then \( (x-c) \) is a factor of \( f(x) \) if and only if \( f(c) = 0 \).

Given \( f(x) = 2x^3 - 5x^2 + 3x - 2 \), which are factors?

\[ a) \quad x + 1 = x - (-1) \quad f(-1) = 2(-1)^3 - 5(-1)^2 + 3(-1) - 2 = -2 - 5 - 3 - 2 = -12 \neq 0 \]

\( \text{NOT A FACTOR} \)

\[ b) \quad x - 2 \quad f(2) = 2(2)^3 - 5(2)^2 + 3(2) - 2 = 16 - 20 + 6 - 2 = 0 \]

\( \text{A FACTOR} \)

\[
\begin{align*}
2x^2 - x + 1 & = \frac{x - 2)(2x^3 - 5x^2 + 3x - 2)}{2x^2 - x + 1} \frac{2x^3 - 4x^2}{-}\frac{x^2 + 3x}{-}\frac{(-x^2 + 2x)}{-x - 2}.
\end{align*}
\]
So \(2x^3 - 5x^2 + 3x - 2 = (2x^2 - x + 1)(x - 2)\)

**Thm. (Descartes' Rule of Signs)** for a poly.

In standard form (descending order of deg).

(a) Number of positive real zeroes of \(f\) either equals the number of variations in sign of non-zero coeffs of \(f\) or an even number less.

**Ex.** \(f(x) = 2x^3 - 5x^2 + 3x - 2\)

So have 3 or 1 positive zeroes.

(b) Number of negative real zeroes of \(f\)

is number of variations in sign of \(f(-x)\) or less than the variation by an even number.
\[ f(x) = 2x^3 - 5x^2 + 3x - 2 \]
\[ f(-x) = 2(-x)^3 - 5(-x)^2 + 3(-x) - 2 \]
\[ = -2x^3 - 5x^2 - 3x - 2 \] \( \quad \) **0 NEGATIVE REAL ROOTS.**

**EX** DISCUSS REAL ZEROS OF

\[ f(x) = (5x^7 - 7x^4 + 2x^3 + 4x^2 + x + 5) \]
\[ \quad + 1 \quad + 1 \]

POSITIVE REAL ZEROS: 4, 2, 0

\[ f(-x) = (-5x^7 - 7x^4 - 2x^3 + 4x^2 - x + 5) \]
\[ \quad + 1 \quad \text{NEGATIVE REAL ZEROS: 1} \]

**RATIONAL ZERO TEST** for a poly or degree 1 or higher of form \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \) (all coeffs are integers) if \( \frac{p}{q} \) (fraction in lowest terms) is a rational zero of \( f \) then \( p \) is a factor of \( a_0 \) and \( q \) a factor of \( a_n \).
\[ f(x) = a_n x^n + \ldots + a_0 = (q x - p) \overbrace{\ldots} \overbrace{0} \]

\[ q(x - \frac{p}{q}) = (q x - p) \]

\[ f(x) = 2x^3 - 8x^2 + 5x - 15 \]

\[ \frac{p}{q} \rightarrow 1, 3, 5, 15 \]

\[ q \rightarrow 1, 2. \]

Possible Rational Factors: \( \pm \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{15}{2} \)

Procedure for Finding Real Zeros (and factoring polynomials)

1. Degrees = Max Number of Real Zeros.
2. Descartes' Rule of Signs: \( \# \text{POS} / \# \text{NEG} \)
3. If Poly has Integer Coeffs' Use Rational Root Test to Determine which Rational Num. to Check.

\[ f(x) = 1x^5 + 5x^4 + 2x^3 - 16x^2 - 8x + 16 \]

\( \# \text{POS REALS: 2 or 0} \) \, \# \text{NEG REALS: 3 or 1} \)
\[ f(-x) = -x^5 + 5x^4 - 2x^3 - 16x^2 + 8x + 16 \]

Possible Rational Zeros: \( \pm 1, \pm 2, \pm 4, \pm 8, \pm \infty \)

Finding Possible Factors of \( f(x) \)

When \( x = 1 \):

\[
\begin{array}{c|cccccc}
    & 1 & 5 & 2 & -16 & -8 & 16 \\
\hline
  1 &   & 1 & 6 & 8 & -8 & -16 \\
\end{array}
\]

Remainder: 0

When \( x = -2 \):

\[
\begin{array}{c|ccccc}
    & 1 & 5 & 2 & -16 & -8 \\
\hline
  -2 & 1 & 4 & 0 & -8 & 0 \\
\end{array}
\]

Remainder: 0

Thus, \( f(x) = (x-1)(x^4 + 6x^3 + 8x^2 - 8x - 16) \)

\( (x+2)(x^3 - 4x^2 - 8) \)
\[ = (x-1)(x+2)(x+2)(x^2+2x-4) \]

\[ x^2 + 2x - 4 = 0 \]

\[ x^2 + 2x + 1 = 4 + 1 \]

\[ (x+1)^2 = 5 \]

\[ x + 1 = \pm \sqrt{5} \]

\[ x = -1 \pm \sqrt{5} \]

**Zeros:** \(-2, -2, -1 + \sqrt{5}, -1 - \sqrt{5}\)

**Positive** \(-2, -2, -1 + \sqrt{5}\)

**Negative** \(-1 - \sqrt{5}\)

3.3 - skip.

3.4 Properties of Rational Functions

of form \( \frac{P(x)}{q(x)} \) where \( P(x), q(x) \) polynomials

**Note:** Domain = \( \{ x \mid q(x) \neq 0 \} \)