Chapter 2 - Linear and Quadratic Functions

2.1 Graph Linear Functions

Definition: A **linear function** is a function of the form $f(x) = mx + b$. The graph of a linear function is a line with slope $m$ and $y$-intercept $(0, b)$. The domain of a linear function is the set of all real numbers. Functions that are not linear are called **nonlinear**.

Theorem: The average rate of change of a linear is equal to the slope: $\frac{\Delta y}{\Delta x} = m$.

Theorem: A linear function $f(x) = mx + b$ is increasing over its domain if its slope $m$ is positive. It is decreasing over its domain if its slope $m$ is negative. It is constant over its domain if its slope is zero.

Examples: Finding a linear function given: 1) slope $m$ and $y$-intercept $b$. 2) slope $m$ and a general point, and 3) two points on the graph of the linear function.

Example (2.1.52): Suppose that a company purchase a machine for $120,000. The company chooses to depreciate the machine using a straight-line method over 10 years.

a) Write a linear model that expresses the value $V$ of the machine as a function of its age $x$.
b) What is the (implied) domain of the function?
c) Graph the linear function.
d) What is the value of the machine after 6 years?
e) When will the machine have a value of $72,000?

2.3 Quadratic Functions and Their Zeroes

Definition: A **quadratic function** is a function of the form $f(x) = ax^2 + bx + c$, where $a$, $b$, and $c$ are real numbers and $a \neq 0$. A **quadratic equation** is an equation equivalent to one of the form $ax^2 + bx + c = 0$. The form $ax^2 + bx + c = 0$ is called the standard form. Such equations are also called **second-degree equations**.

Examples: Find the zeroes of a quadratic function by: 1) factoring. 2) using the square root method. 3) completing the square.

Quadratic formula: The solution(s) to the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note that if the discriminant $b^2 - 4ac < 0$, the equation has no real solution.
Solving equations that are quadratic in type: By substituting for a quantity, equations can be transformed into quadratic equations.

example (2.2 66): Find the zeroes of \( f(x) = (2x + 5)^2 - (2x + 5) - 6 \)

2.4 Properties of Quadratic Functions

Notation: The graph of a quadratic function is a parabola; it can open up or open down. The turning point (high point or low point) is called the vertex. The vertical line passing through the vertex is the line of symmetry, or axis of symmetry.

Alternate form for a quadratic function: \( f(x) = ax^2 + bx + c = a(x - h)^2 + k \). Note that with this form, the vertex is at \((h, k)\), and the axis of symmetry is the line \( x = h \).

In the standard form \( f(x) = ax^2 + bx + c \), the vertex is \( \left( -\frac{b}{2a}, f\left( -\frac{b}{2a} \right) \right) \), and the axis of symmetry is the vertical line \( x = -\frac{b}{2a} \). Using either form, the parabola opens up if \( a > 0 \) and opens down if \( a < 0 \).

Steps for graphing a quadratic equation are summarized on page 155 of text.

example: Sketch the graph of \( f(x) = 2x^2 + 8x - 10 \)

Find a quadratic function given its vertex and one other point: Given the vertex \((h, k)\) and another point on the parabola, use the form \( f(x) = a(x - h)^2 + k \). The constants \( h \) and \( k \) are already known. In order to find the one remaining constant \( a \), substitute the coordinates of the point in for \( x \) and \( f(x) \).

example: Find the equation of the parabola with vertex \((2, 5)\) that passes through the point \((6, -3)\).

Note: The maximum value of a quadratic function that opens down is \( f\left( -\frac{b}{2a} \right) \). The minimum value of a quadratic function that opens up is \( f\left( -\frac{b}{2a} \right) \). So the vertex is the point on the graph at which a quadratic function attains its extreme value.
2.5 Inequalities Involving Quadratic Functions

Solving an inequality involving a quadratic function: eg. \( f(x) = ax^2 + bx + c \geq 0 \)

1) Find the zeroes of \( f(x) = ax^2 + bx + c \); note that these \( x \)-values break the \( x \)-axis into three intervals. Choose test points in each interval to determine whether the function is positive or negative. If the inequality is not strict, include the endpoints in the solution.

2) Sketch the graph of \( f(x) = ax^2 + bx + c \); note where the function is above and below the \( x \)-axis. Choose the appropriate interval(s) where \( f(x) \) is the desired quantity.

example: Solve \( x^2 + 2x - 15 \geq 0 \).

example: Solve \( 3x^2 < 10 - x \)

2.6 Building Quadratic Models From Verbal Descriptions and from Data

Often a quadratic function is used in problems involving optimization, i.e. finding a maximum or a minimum value of a quantity.

example: A rancher wants to build a corral next to a straight river, so she will only have to build a rectangular fence on three sides. If she has three hundred feet of fence material, find the dimensions of the corral that will maximize the area of the corral.

example (example 4 in text): The Golden Gate Bridge has towers that are 746 feet tall above the water and 4200 feet apart. The roadway on the bridge is 220 feet above the water and is suspended from two cables that are parabolic in shape. If these cables touch the roadway in the center of the bridge (halfway between the two towers), find the height of the cables above the road 1000 feet from the center of the bridge.