Chapter 1 - Functions and Their Graphs

1.1 Functions

**Definition:** A **relation** is a correspondence between two sets.

**Definition:** A **function** from a set \(X\) to another set \(Y\) is a relation that associates to each element in \(X\) exactly one element in \(Y\). The set \(X\) is called the **domain** of the function, and the set of outputs in \(Y\) is called the **range**. For \(x \in X\), the corresponding element \(y \in Y\) is called the **image** of \(x\).

Examples:

<table>
<thead>
<tr>
<th>input</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>output</td>
<td>4</td>
<td>7</td>
<td>10</td>
<td>13</td>
<td>16</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>input</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>output</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(x)</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

\{ (1, 5), (2, 8), (3, 4), (4, 9) \}

Notation: value in domain = \(x\) = input = independent variable
value in range = \(y = f(x)\) = output = dependent variable

Convention: unless specified, domain \(D\) is largest possible subset of \(\mathbb{R}\)

Examples: \(y = \sqrt{3x + 12}\) \(D = \)

\(s = \frac{t - 1}{t^2 + 3t - 10}\) \(D = \)
1.2 The Graph of a Function

**Definition:** A graph of a function $f$ in the $xy$-plane consists of those points $(x, y)$ such that $x$ is in the domain of $f$ and $y = f(x)$.

$$(x, y) = (x, f(x))$$

**Vertical line test:** A graph in the $xy$-plane represents $y$ as a function of $x$ if any vertical line intersects the graph in at most one point.

Examples: Graphs of functions--domain & range

1) $f(x) = |x|$
2) $f(x) = -\sqrt{x}$
3) $y = \sqrt{25 - x^2}$

1.3 Properties of Functions

A function is **increasing** on an open interval $I$ if, for any choice of $x_1$ and $x_2$ with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

A function is **decreasing** on an open interval $I$ if, for any choice of $x_1$ and $x_2$ with $x_1 < x_2$, we have $f(x_1) > f(x_2)$.

A function is **constant** on an open interval $I$ if, for all choices of $x \in I$, the values of $f(x)$ are equal.

Points on a graph where the graph changes from rising to falling (or vice versa) are called turning points.
A function $f$ has a **local maximum** at $c$ if there is an open interval $I$ containing $c$ so that for all $x$ in $I$, $f(x) \leq f(c)$. $f(c)$ is the **local maximum**.

A function $f$ has a **local minimum** at $c$ if there is an open interval $I$ containing $c$ so that for all $x$ in $I$, $f(x) \geq f(c)$. $f(c)$ is the **local minimum**.

We say that the **absolute maximum of** $f$ on an interval $I$, if there exists a value $u$ such that for all $x$ in $I$, $f(x) \leq f(u)$. $f(u)$ is the **absolute maximum**.

We say that the **absolute minimum of** $f$ on an interval $I$, if there exists a value $u$ such that for all $x$ in $I$, $f(x) \geq f(u)$. $f(u)$ is the **absolute minimum**.

Note: The absolute maximum or absolute minimum may not exist.

Note: Calculus is usually needed to find exact values of these coordinates.

**Definition**: If $a$ and $b$, $a \neq b$, are in the domain of a function $f$, the **average rate of change** of $f$ from $a$ to $b$ is defined as

$$\frac{\Delta y}{\Delta x} = \frac{f(b)-f(a)}{b-a}$$

Example: Find average rate of change for $f(x) = x^3$ on the interval $[1, 3]$.

Note: The average rate of change of a function $f$ from $a$ to $b$ is the slope of the **secant line** containing the points $(a, f(a))$ and $(b, f(b))$.

**1.4 Library of Functions; Piecewise-defined Functions**

Functions that you should know:

- **Constant functions**: $f(x) = b$
- **Identity function**: $f(x) = x$
- **Square function**: $f(x) = x^2$
- **Cube function**: $f(x) = x^3$
- **Square root function**: $f(x) = \sqrt{x}$
- **Cube root function**: $f(x) = \sqrt[3]{x}$
- **Reciprocal function**: $f(x) = \frac{1}{x}$
- **Absolute value function**: $f(x) = |x|$
Greatest integer function: \( f(x) = \|x\| = \) largest integer less than or equal to \( x \).

**Piecewise defined functions:** functions with different equation for different parts on its domain

example: \( f(x) = \begin{cases} 
  x^2 & x \leq 3 \\
  2x + 1 & x > 3 
\end{cases} \)

**1.6 Mathematical Models: Building Functions**

Functions often need to be constructed from information given in a problem.

example (1.6.3a): Express the distance from a point \( P = (x, y) \) on the graph of \( y = \sqrt{x} \) to the point \((1, 0)\) as a function of \( x \).

example (1.6.8a): Express the area \( A \) of the rectangle that is inscribed in the semicircle of radius 2 centered at the origin as a function of \( x \).