4.2 One-to-One Functions; Inverse Functions

Definition: A function \( f \) is one-to-one if any two different inputs in the domain correspond to two different outputs in the range. That is, if \( x_1 \neq x_2 \), then \( f(x_1) \neq f(x_2) \).

<table>
<thead>
<tr>
<th>input</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>output</td>
<td>4</td>
<td>7</td>
<td>10</td>
<td>13</td>
<td>16</td>
</tr>
</tbody>
</table>

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<thead>
<tr>
<th>input</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>output</td>
<td>1</td>
<td>5</td>
<td>8</td>
<td>5</td>
</tr>
</tbody>
</table>

\[
\begin{array}{c|c|c|c|c}
 x & 1 & 3 & 5 & 7 \\
 y & 8 & 8 & 8 & 8 \\
\end{array}
\]

\{ (1, 5), (2, 8), (3, 6), (4, 9) \}

Theorem (Horizontal line test): If every horizontal line intersects the graph of a function \( f \) in at most one point, then \( f \) is one-to-one.

Theorem: A function that is increasing on an interval \( I \) is a one-to-one function on \( I \). A function that is decreasing on an interval \( I \) is a one-to-one function on \( I \).

Definition: Suppose that \( f \) is a one-to-one function. Then corresponding to each \( x \) in the domain of \( f \), there is exactly one \( y \) in the range; and corresponding to each \( y \) in the range of \( f \), there is exactly one \( x \) in the domain. The correspondence from the range of \( f \) to the domain of \( f \) is called the inverse function of \( f \). The symbol \( f^{-1} \) is used to denote the inverse of \( f \).

Examples: use one-to-one functions above.

Notes: 1) Domain of \( f^{-1} = \) Range of \( f \)  Range of \( f^{-1} = \) Domain of \( f \)

2) \( f[f^{-1}(x)] = x \) for each \( x \) in the domain of \( f^{-1} \)
   \( f^{-1}[f(x)] = x \) for each \( x \) in the domain of \( f \)

Theorem: The graph of a one-to-one function of \( f \) and the graph of its inverse \( f^{-1} \) are symmetric with respect to the line \( y = x \).

Given a function \( f \), to find the inverse function \( f^{-1} \) (if it exists):

0) Write \( y = f(x) \)
1) Interchange values for \( y \) and \( x \): \( x \leftrightarrow y \)
2) Solve new equation for \( y \): \( y = [\) new formula in terms of \( x \) \( ] = f^{-1}(x) \).
3) Check the result by showing \( f[f^{-1}(x)] = x \) and \( f^{-1}[f(x)] = x \)

Examples: \( f(x) = 2x - 6 \) \( g(x) = \frac{x+3}{2x-5} \) \( h(x) = x^2 \) \( x \geq 0 \)