3.2 Taylor's Theorem

Recall (from single variable calculus): if \( f(x) \) is infinitely differentiable near \( x = x_0 \), then

\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!(x - x_0)^2} + \ldots + \frac{f^{(k)}(x_0)}{k!(x - x_0)^k} + R_k(x_0)
\]

set \( x = x_0 + h \), and the above equation becomes

\[
f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \frac{f'''(x_0)}{6}h^3 + \ldots + \frac{f^{(k)}(x_0)}{k!}h^k + R_k(x_0, h)
\]

where \( R_k(x_0, h) = \int_{x_0}^{x_0+h} \frac{(x_0+h-\tau)}{k!} f^{(k+1)}(\tau)d\tau \).

Claim: \( \lim_{h \to 0} \frac{R_k(x_0,h)}{h^k} = 0 \)

Proof: By the Fundamental Theorem of Calculus, \( \int_{x_0}^{x_0+h} f'(\tau)d\tau = f(x_0 + h) - f(x) \)
So \( f(x_0 + h) = f(x_0) + \int_{x_0}^{x_0+h} f'(\tau)d\tau \). Use IBP to integrate the integral.

\[
u = f'(\tau) ; \ dv = d\tau \quad \Rightarrow \quad du = f''(\tau)d\tau ; \ v = -(x_0 + h - \tau) [\text{can use any antiderivative}]
\]

So \( f(x_0 + h) = f(x_0) + f'(x_0)[-(x_0 + h - \tau)] \bigg|_{x_0}^{x_0+h} + \int_{x_0}^{x_0+h} f''(\tau)(x_0 + h - \tau)d\tau \)

\[
f(x_0 + h) = f(x_0) + f'(x_0)(h)[0] + f''(x_0)[h] + \int_{x_0}^{x_0+h} f''(\tau)(x_0 + h - \tau)d\tau \)

IBP again: \( u = f''(\tau) ; \ dv = (x_0 + h - \tau)d\tau \quad \Rightarrow \quad du = f'''(\tau)d\tau ; \ v = -\frac{1}{2}(x_0 + h - \tau)^2 \)

\[
\int_{x_0}^{x_0+h} f''(\tau)(x_0 + h - \tau)d\tau = f''(\tau)[-\frac{1}{2}(x_0 + h - \tau)^2] \bigg|_{x_0}^{x_0+h} + \frac{1}{2} \int_{x_0}^{x_0+h} f'''(\tau)(x_0 + h - \tau)^3d\tau
\]

So \( f(x_0 + h) = f(x_0) + f'(x_0)(h) + \frac{1}{2} f''(x_0)(h)^2 + \frac{1}{2} \int_{x_0}^{x_0+h} f'''(\tau)(x_0 + h - \tau)^3d\tau \)

Continuing with IBP yields \( |R_k(x_0, h)| = \left| \int_{x_0}^{x_0+h} \frac{(x_0+h-\tau)^k}{k!} f^{(k+1)}(\tau)d\tau \right| \leq \frac{|h|^{k+1}}{k!} M \)

where \( M = \max |f^{(k+1)}(\tau)| \) for \( \tau \in (x_0, x_0 + h) \); so \( \lim_{h \to 0} \frac{R_k(x_0,h)}{h^k} \leq \lim_{h \to 0} \frac{|h|}{k!} M = 0 \)
Taylor's Theorem for Many Variables

If \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is differentiable at \( x_0 \), define \( R_1(x_0, h) = f(x_0 + h) - f(x_0) - [Df(x_0)](h) \)
so \( f(x_0 + h) = f(x_0) + [Df(x_0)](h) + R_1(x_0, h) \); by the definition of differentiability, we have \( \frac{R_1(x_0, h)}{\|h\|} \rightarrow 0 \) as \( h \rightarrow 0 \). This proves the following theorem:

Theorem (First Order Taylor Formula): Let \( f : U \subset \mathbb{R}^n \rightarrow \mathbb{R} \) be differentiable at \( x_0 \in U \).

Then \( f(x_0 + h) = f(x_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(x_0) + R_1(x_0, h) \) where \( \frac{R_1(x_0, h)}{\|h\|} \rightarrow 0 \) as \( h \rightarrow 0 \).

Theorem (Second Order Taylor Formula): Let \( f : U \subset \mathbb{R}^n \rightarrow \mathbb{R} \) be have continuous derivatives of third order (class \( C^3 \)). Then

\[
\begin{aligned}
f(x_0 + h) &= f(x_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(x_0) + \frac{1}{2} \sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) + R_2(x_0, h),
\end{aligned}
\]

where \( \frac{R_2(x_0, h)}{\|h\|^2} \rightarrow 0 \) as \( h \rightarrow 0 \). [Note that the second sum has \( n^2 \) terms]

Proof: in text. We will only be using Taylor's formula, and to find approximations.

example (3.2.4): Determine the second order Taylor formula for \( f(x, y) = \frac{1}{x^2 + y^2 + 1} \) about \((0, 0)\). Note \( \frac{\partial f}{\partial x} = \frac{-2x}{(x^2 + y^2 + 1)^2} \), \( \frac{\partial f}{\partial y} = \frac{-2y}{(x^2 + y^2 + 1)^2} \), \( \frac{\partial^2 f}{\partial x^2} = \frac{-2(x^2 + y^2 + 1) + 4x}{(x^2 + y^2 + 1)^3} \),

\[
\begin{aligned}
\frac{\partial^2 f}{\partial y^2} &= \frac{-2(x^2 + y^2 + 1) + 4y}{(x^2 + y^2 + 1)^3},
\end{aligned}
\]

and \( \frac{\partial^2 f}{\partial x \partial y} = \frac{8xy}{(x^2 + y^2 + 1)^3} \).

example (example 3 in text): Find the first and second order Taylor approximations for \( f(x, y) = \sin(xy) \) at the point \((1, \frac{\pi}{2})\).