2.4 Introduction to Paths and Curves

Definition: A path \( C \) in \( \mathbb{R}^n \) is a map \( \mathbf{c} : [a, b] \in \mathbb{R} \to \mathbb{R}^n \)

if \( n = 2 \), the path \( \mathbf{c}(t) = (x(t), y(t)) \) is a planar curve.

if \( n = 3 \), the path \( \mathbf{c}(t) = (x(t), y(t), z(t)) \) is a space curve.

\( \mathbf{c}(a) \) is the initial point of the curve, and \( \mathbf{c}(b) \) is the terminal point.

examples: 1) \( \mathbf{c}(t) = (3 + 2t, 2 - 3t, 5 + t) \) is a line in \( \mathbb{R}^3 \).

2) \( \mathbf{c}(t) = (cos \ t, \ sin \ t) \) is the unit circle in \( \mathbb{R}^2 \).

3) \( \mathbf{c}(t) = (t - \ sin \ t, \ 1 - \ cos \ t) \) is a cycloid in \( \mathbb{R}^2 \).

Definition: If \( \mathbf{c}(t) \) is a path, and it is differentiable, then we call \( \mathbf{c} \) a differentiable path. The velocity of \( \mathbf{c} \) at time \( t \) is defined by \( \mathbf{c}'(t) = \lim_{h \to 0} \frac{\mathbf{c}(t+h)-\mathbf{c}(t)}{h} \). It is customary to draw the vector \( \mathbf{c}'(t) \) with its initial point (tail) at \( \mathbf{c}(t) \). The speed of the path \( \mathbf{c}(t) \) is \( s = \|\mathbf{c}'(t)\| \). If \( \mathbf{c}(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3 \), then \( \mathbf{c}'(t) = (x'(t), y'(t), z'(t)) \).

Notes: 1) The vector \( \mathbf{c}'(t) \) is a vector tangent to the curve \( C \) at the point \( \mathbf{c}(t) \).

2) The derivative matrix \( \mathbf{Dc}(t) \) is an \( n \times 1 \) column vector with entries \( x'_1(t), x'_2(t), \ldots, x'_n(t) \)

examples: Compute the tangent vector to:

1) \( \mathbf{c}(t) = \left( t^2, t^3 - 1, t^4 - 1 \right) \)

2) \( \mathbf{c}(t) = (cos \ t, \ sin \ t, \ t) \) [this is a helix]

Definition: If \( \mathbf{c}(t) \) is a path, and if \( \mathbf{c}'(t_0) \neq 0 \), the equation of its tangent line at the point \( \mathbf{c}(t_0) \) is \( \mathbf{l}(t) = \mathbf{c}(t_0) + (t - t_0)\mathbf{c}'(t_0) \). If \( C \) is the curve traced out by \( \mathbf{c} \), then the line traced out by \( \mathbf{l} \) is the tangent line to the curve \( C \) at \( \mathbf{c}(t_0) \).

example: Find the equation of the tangent line to the curve given by \( \mathbf{c}(t) = (e^t, cos(\pi t), ln \ t) \) at the point when \( t = 1 \). Find the point on the tangent line when \( t = 4 \).