2.6 GRADIENTS \[\frac{1}{3}\text{DIRECTIONAL DERIVATIVES}\]

\[f: \mathbb{R}^3 \to \mathbb{R}\]

GRADIENT of \(f(x, y, z) = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})\)

DEF (DIRECTIONAL DERIVATIVE) IF \(f: \mathbb{R}^3 \to \mathbb{R}\)

THE DIRECTIONAL DERIVATIVE of \(f\) at \(\vec{x}\) in DIRECTION of A UNIT VECTOR \(\vec{v}\) IS GIVEN BY

\[
\frac{d}{dt}\left[f(\vec{x} + t\vec{v})\right]_{t=0} = \nabla f(\vec{x}) \cdot \vec{v}
\]

DOT PRODUCT

So \(\nabla f(\vec{x}) \cdot \vec{v} = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) \cdot (v_1, v_2, v_3)\)

NOTE: \(\vec{c}(t)\) IS A PATH IN \(\mathbb{R}^3\) THEN \(\vec{c}(0) = \vec{x}\) AND \(\vec{c}'(0) = \vec{v} \leftarrow \text{MAY NOT BE A UNIT VECTOR.}\)
\[
\frac{d}{dt} f(\bar{c}(t)) \Bigg|_{t=0} = \nabla f(\bar{c}(0)) \cdot \bar{c}'(0) = \nabla f(x) \cdot \bar{v}
\]

WANT A UNIT VECTOR

\text{(ex. 1) } \text{IF } f(x, y, z) = x^2 + xy^2z + z^3 \text{ FIND DIRECTED DERIVATIVE OF } f \text{ AT } (1, 2, 3) \text{ IN DIRECTION OF } \bar{v} = \frac{1}{3}(2, -2, 1)

\nabla f = \left(2x + yz^2, xz^2, xy + 3z^2\right) \bigg|_{(1, 2, 3)}

= (2(1) + (2)(3), (1)(3), (1)(2) + 3(3)^2)

= (8, 3, 29)

\text{D.D. } = \nabla f \cdot \bar{v} = (8, 3, 29) \cdot \frac{1}{3}(2, -2, 1)

= \frac{1}{3} \left(16 + (-6) + 29\right) = \frac{1}{3}(39) = 13
Ex. 2) \( f(x, y, z) = xy^2z^3 \) find \( \text{D.D. at} \) \((3, 2, 1) \) in direction \((1, 3, -5)\) not a unit vector

Unit vector:
\[
\vec{V} = \frac{1}{\sqrt{35}} (1, 3, -5)
\]

\[
\nabla f = (y^2z^3, x[2yz^3], xy^2[3z^2]) \bigg|_{(3, 2, 1)} = (4, 12, 36)
\]

\[
\text{D.D.} = (4, 12, 36) \cdot \left( \frac{1}{\sqrt{35}} (1, 3, -5) \right)
\]

\[
= \frac{1}{\sqrt{35}} (4 + 36 + (-180)) = \frac{-140}{\sqrt{35}}
\]

\[
= -\frac{4 \cdot 35}{\sqrt{35}} = -4\sqrt{35}
\]

Note: if \( f: \mathbb{R}^2 \to \mathbb{R} \) then \( \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \)

If \( \vec{V} = (a, b) \) (unit vector) dir. der of \( f = \nabla f \cdot \vec{V} \)
NOTE: THE GRADIENT $\nabla f(x_0)$ IS NORMAL TO LEVEL SET OF $f$ THROUGH $(x_0, f(x_0))$

$z = f(x_0, y_0)$

LEVEL CURVES

$z = k$

THM: Assume $\nabla f(x) \neq 0$, then

$\nabla f$ points in direction of maximum increase.

$D.D. = \nabla f(x) \cdot \nabla = \left| \frac{\nabla f(x)}{||\nabla f(x)||} \right| \cdot \cos \theta$

$D.D.$ is maximum when $\cos \theta = 1$ i.e. $\theta = 0$

i.e. $\nabla$ is same direction as $\nabla f(x)$

Max $D.D.$ is $|\nabla f(x)|$
**THM:** Let \( f: \mathbb{R}^3 \rightarrow \mathbb{R} \) be a \( C^1 \) map, and let \( (x_0, y_0, z_0) = \mathbf{x}_0 \) be on level surface \( f(x, y, z) = k \) then \( \nabla f(x_0, y_0, z_0) \) is normal to level surface \( f(x, y, z) = k \).

**DFN** Let \( S \) be surface \( f(x, y, z) = k \) the tangent plane to \( S \) at \( (x_0, y_0, z_0) \) \([ons]\)
is defined by:

\( \nabla f(x_0, y_0, z_0) \cdot (x-x_0, y-y_0, z-z_0) = 0 \)

ex given \( x^2 - 4y^2 + z^2 = 10 \) find.

a) normal line to surface at \((3, -1, 5)\)

b) tangent plane to surface at \((3, -1, 5)\)

\( \nabla f = (2x, -8y, 1) \)

\( \nabla f \bigg|_{(3, -1, 5)} = (6, 8, 1) \)

a) \( x = 3 \pm 6t \)
\( y = -1 + 8t \)
\( z = 5 + 4t \)

b) \((6, 8, 1) \cdot (x-3, y+1, z-5) = 0 \)

\( 6x + 8y + z = 15 \)

\( 6x - 18 + 8x + 8 + 2 - 5 = 0 \)

\( 6x + 8y + 2 - 15 = 0 \)