14.8 Lagrange Multipliers

Note that on a contour map (i.e. the graphs of level curves) of a surface \( z = f(x, y) \), a path \((g(x, y) = k)\) on the map achieves both maximum and minimum values (of the function \( f \)) when the path is tangent to a level curve. Since the tangent lines to each curve are the same, so are the normal directions; so we have normal vectors to each curve parallel. Hence

\[
\nabla f(x, y) = \lambda \cdot \nabla g(x, y)
\]

**Method of Lagrange multipliers:** To find the maximum and minimum values of a function \( f(x, y, z) \) subject to the constraint \( g(x, y, z) = k \) [here we assume that extreme values of \( f \) exist and \( \nabla g \neq 0 \) on the surface ]:

a. Find all values \( x, y, z \) and \( \lambda \) such that

\[
\nabla f(x, y, z) = \lambda \cdot \nabla g(x, y, z) \quad \text{and} \quad g(x, y, z) = k
\]

b. Evaluate \( f \) at all of the points \((x, y, z)\) that result from step a. The largest of those values is the maximum value of \( f \) and the smallest is the minimum value of \( f \).

**Example (14.8.5):** Find the maximum and minimum values of \( f(x, y) = y^2 - x^2 \) subject to the constraint \( \frac{1}{4} x^2 + y^2 = 1 \).

**Example (14.8.9):** Find the max/min values of \( f(x, y, z) = xyz \) subject to the constraint \( g(x, y, z) = x^2 + 2y^2 + 3z^2 = 6 \).

**Example (14.8.38/14.7.48)** Use Lagrange multipliers to find the dimensions of the rectangular box with largest volume if the total surface area is 64 cm².

**Lagrange multipliers with two constraints**

Maximize (or minimize): \( f(x, y, z) \) subject to: \( g(x, y, z) = k \) and \( h(x, y, z) = c \)

Solve: \( \nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z) \). This yields a system of 5 equations, with 5 unknowns \((x, y, z, \lambda, \mu)\).

**Example:** Find the extreme values of \( f(x, y, z) = xy + yz \) subject to \( g(x, y, z) = xy = 1 \) and \( h(x, y, z) = y^2 + z^2 = 2 \).