14.7 Maximum; Minimum Values

Recall: \( y = f(x) \)

Critical Point: \( f'(x) = 0 \) (or D.E.E.)

Consider \( z = f(x, y) \)

Define

Local Max at \((a, b)\): \( f(a, b) \geq f(x, y) \) if \((x, y)\) is near \((a, b)\)

Local Min at \((a, b)\): \( f(a, b) \leq f(x, y) \) for \((x, y)\) near \((a, b)\)

Absolute Extremes: Largest/Smallest Function Values in Domain.

This if \( f \) has a local extreme at \((a, b)\) and if \( f_x(a, b) \) or \( f_y(a, b) \) exist, then \( f_x(a, b) = f_y(a, b) = 0 \).
**Definition:** Point \((a, b)\) is a **critical point** for \(f\) if \(f_x(a, b) = 0\) and \(f_y(a, b) = 0\) (or if one (or both) does).

**Example:** Find critical point(s) for:

a) \(f(x, y) = x^2 + y^2 - 6x + 10y + 45\)

\[
\begin{align*}
f_x(x, y) &= 2x - 6 = 0 \\
f_y(x, y) &= 2y - 10 = 0
\end{align*}
\]

At \((3, 5)\)

Note: \(f_{xx}(x, y) = 2x - 6 + \frac{9}{y} + y^2 - 10y + 25 + 45 = 0\)

\[z = f(x, y) = (x - 3)^2 + (y - 5)^2 + 11\]

b) \(f(x, y) = x^3 y + 12x^2 - 8y\)

\[
\begin{align*}
f_x(x, y) &= 3x^2 y + 24x = 0 \\
f_y(x, y) &= x^3 - 8 = 0 \rightarrow x = 2
\end{align*}
\]

Critical point \((2, -4)\)
SECOND DERIVATIVE TEST: Suppose second partial der. $f_{xx}$, $f_{yy}$ are continuous on a disk w/ center $(a,b)$, $(a,b)$ CRITICAL PT or $f(x,y)$. Let $D = D(a,b) = f_{xx}(a,b) \cdot f_{yy}(a,b) - [f_{xy}(a,b)]^2$, THEN:

a) if $D > 0$ and $f_{xx}(a,b) > 0$ then $f(a,b)$ is a local minimum.

b) if $D > 0$ and $f_{xx}(a,b) < 0$ then $f(a,b)$ is local max.

c) if $D < 0$ then $f(a,b)$ is neither a local max nor a local min (SADDLE POINT).

Ex. Find & classify critical pt for:

$f(x,y) = 2x^3 + xy^2 + 8x^2 + y^2$

$f_x(x,y) = 6x^2 + y^2 + 10x = 0$

$f_y(x,y) = 2xy + 2y = 0$

$2y(x+1) = 0 \rightarrow y = 0$ \(\text{(A)}\)

or

$y = -1$ \(\text{(B)}\)$
1F (A) \[ 6x^2 + (0)^2 + 10x = 0 \]  
\[ 6x^2 + 10x = 0 \]  
\[ 2x(3x + 5) = 0 \rightarrow x = 0, x = -\frac{5}{3} \]

(CRITICAL PTS)

\[ (0,0) \text{ j } \left(-\frac{5}{3},0\right) \]

1F (B) \[ 6(-1)^2 + y^2 + 10(-1) = 0 \]  
\[ y^2 - 4 = 0 \rightarrow y = \pm 2 \]

\[ (-1,2) \text{ j } (-1,-2) \]

(CRITICAL PTS)

\[ f_{xx} = 12x + 10 \]

\[ f_{yy} = 2x + 2 \]

\[ f_{xy} = 2y = f_{yx} \]

\[
\begin{array}{c|ccc}
(0,0) & (-\frac{5}{3},0) & (-1,\pm 2) \\
\hline
f_{xx}(0,0) & 10 & -10 & -2 \\
D & 20 & 0 & 0 & -16
\end{array}
\]

\[ f(0,0) = 0 \text{ (LOCAL MIN)} \]

\[ f\left(-\frac{5}{3},0\right) = -\frac{250}{27} + \frac{125}{9} \text{ (LOCAL MAX)} \]

\[ f(-1\pm 2) \text{ SADDLE POINTS} \quad (f_{xy})^2 \]

\[ \text{NOTE: } D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2 \]
THEOREM: If \( f \) is continuous on a closed, bounded set \( D \) in \( \mathbb{R}^2 \), then \( f \) attains both an absolute max and absolute min. value on \( D \).

**Strategy:**
1. Find values of \( f \) at critical points in \( D \).
2. Find extreme values of \( f \) on the boundary of \( D \) (often, single variable problems).
3. Largest is \( \max \); smallest is \( \min \).

\[
f(x,y) = 2x + 5y \quad \text{on} \quad x^2 + y^2 \leq 1
\]

\[
f = 2\cos t + 5\sin t
\]

Boundary: \( x^2 + y^2 = 1 \)
\[x = \cos t, \quad 0 \leq t \leq 2\pi\]
\[y = \sin t\]

\[f(x,y) = 3 + xy - x - 2y \quad \text{on} \quad D \text{: Triangle w/ vertices at } (1,0); (5,0); (1,4)\]

\[\text{Min.}\]
\[\text{Max}\]
\[ f_x = y - 1 = 0 \]
\[ f_y = x - 2 = 0 \quad \text{\textcircled{2,1}} \]
\[ f(2,1) = 1 \]

1. \[ f(x,0) = 3 - x \quad \frac{df}{dx} = -1 \neq 0 \]
\[ f(5,0) = -2 \]

2. \[ f(1,y) = 3 + y - 1 - 2y = 2 - y \]
\[ f(1,0) = 2 \]
\[ f(1,4) = -2 \]

3. \[ f(x,5-x) = 3 + x(5-x) - x - 2(5-x) \]
\[ = 3 + 5x - x^2 - x - 10 + 2x \]
\[ = -x^2 + 6x - 7 \]
\[ -2x + 6 = 0 \quad \text{\textcircled{3,2}} \]
\[ x = 3 \]
\[ f(3,2) = 2 \]

\[ f_{\text{max}} = 2 \]
\[ f_{\text{min}} = -2 \]