14.6 **Directional Derivatives**; the Gradient Vector (Cont'd)

**Directional Derivative** at \((x_0, y_0)\) in direction of (unit) vector \(\mathbf{v}\) is:

\[
D_{\mathbf{v}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h\alpha, y_0 + h\beta) - f(x_0, y_0)}{h}
\]

\(\mathbf{v} = \langle \alpha, \beta \rangle\)

Then if \(f\) is differentiable, then \(D_{\mathbf{v}} f(x_0, y_0)\) for any (unit) \(\mathbf{v}\) exists, and

\[
D_{\mathbf{v}} f(x_0, y_0) = f_x(x_0, y_0) \alpha + f_y(x_0, y_0) \beta
\]

**Proof (in written notes, and in text)**

**Example**

Let \(f(x, y) = \frac{x}{y}\). Find \(D_{\mathbf{v}} f(4, 1)\) where \(\mathbf{v} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle\).

\[
f_x = \frac{1}{y}; \quad f_y = x \left[ -y^{-2} \right] = \frac{-x}{y^2}
\]

So

\[
D_{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} f(4, 1) = \left( \frac{1}{y} \right) \cdot \frac{1}{\sqrt{2}} + \left( \frac{-x}{y^2} \right) \cdot \frac{1}{\sqrt{2}} = (4, 1)
\]
\[
\frac{1}{(1)^2 \sqrt{2}} - \frac{4}{(1)^2 \sqrt{2}} = \frac{-3}{\sqrt{2}}
\]

**DFN** IF \( f \) IS A FCN OF \((x, y)\) THE GRADIENT OF \( f \) IS VECTOR FUNCTION, DENOTED \( \nabla f \)

\[
\nabla f = \langle f_x(x_0, y), f_y(x, y) \rangle
\]

**NOTE** \[
D_n f(x_0) = f_x(x_0, y) \cdot a + f_y(x, y) \cdot b = \nabla f \cdot \vec{a}
\]

\[
\begin{align*}
\text{ex} & \quad f(x, y) = e^{x^2 \ln(y)} \quad \text{FIND } \nabla f \\
\nabla f & = \langle f_x(x, y), f_y(x, y) \rangle \\
& = \langle 2x \ln(y), \frac{x^2}{y} \rangle
\end{align*}
\]

VIEW THIS FUNCTION IN \( \mathbb{R}^2 \)

\[
\begin{align*}
\text{ex} & \quad f(x, y) = x^3 y^2 + y \tan^{-1}(x) \\
\text{FIND DIR. DERIV. OF } f \text{ AS } (1,3) \\
\text{IN DIRECTION } \vec{v} & = \langle 4, -5 \rangle \\
\vec{u} & = \frac{\nabla f}{\|
\nabla f\|} = \frac{\langle 4, -5 \rangle}{\sqrt{4^2 + (-5)^2}} = \langle 4, -5 \rangle
\end{align*}
\]
\[ \nabla f = \left\langle 3x^2y^2 + y\left[ \frac{1}{1 + (xy)^2} \right], 2x^3y + \tan^{-1} x \right\rangle \]

\[ = \left\langle 3(1)^2(3)^2 + (3) \frac{1}{1 + (1)^2}, 2(1)^3(3) + \tan^{-1} (1) \right\rangle \]

\[ = \left\langle 27 + \frac{3}{2}, 6 + \frac{\pi}{4} \right\rangle = \left\langle \frac{24}{4} \right\rangle \]

\[ \left( \frac{4}{\sqrt{41}} \right) \cdot \left( \frac{4}{\sqrt{41}} \right) \]

\[ = \left\langle \frac{57}{2}, \frac{24 + 14}{4} \right\rangle = \left\langle \frac{456}{4} \right\rangle \]

\[ = \frac{1}{\sqrt{41}} \left( 114 + \frac{456 - 120 - \pi}{4} \right) = \frac{114 - \frac{456 - 120 - \pi}{4}}{\sqrt{41}} \]

\[ = \frac{386 - \pi}{4\sqrt{41}} \]

**FUNCTIONS WITH THREE VARIABLES**

**DEF** M THE DIRECTIONAL DERIVATIVE OF \( f(x, y, z) \) IN DIRECTION OF \( \left( \text{UNIT} \right) = \left\langle a, b, c \right\rangle \) AS \( (x_0, y_0, z_0) \)

\[ D_{\mathbf{v}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + h\mathbf{v}, y_0 + h\mathbf{w}, z_0 + h\mathbf{z}) - f(x_0, y_0, z_0)}{h} \]
\[ x_0 = (x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + h\bar{w}) - f(x_0)}{h} \]

**NOTE:**

\[ D_{\bar{w}} f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0) \bar{a} + f_y(x_0, y_0, z_0) \bar{b} + f_z(x_0, y_0, z_0) \bar{c} = \nabla f(x_0, y_0, z_0) \cdot \bar{w} \]

WHERE \[ \nabla f(x_0, y_0, z_0) = \left< f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0) \right> \]

\[ \frac{\partial f(x, y, z)}{\partial x} = (\cos x) e^{y^2/2}, \text{ \text{ FIND } D_{\bar{w}} f(\pi/2, 0, 3) \text{ \text{ IN DIRECTION } \bar{w} = \left< \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right>\text{.}} \]

\[ \nabla f(x_0, y_0, z_0) = \left< -\sin x \right> e^{y^2/2}, \cos x \left[ z e^{y^2/2} \right], \cos x \left[ 2 ye^{y^2/2} \right] \]

\[ \nabla f(\pi/2, 0, 3) = \left< -1, 0, 0 \right> \]

So \[ D_{\left< \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right>} f(\pi/2, 0, 3) = \left< -1, 0, 0 \right> \cdot \left< \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right> = \left< -\frac{1}{3} \right> \]

**MAXIMIZING DIRECTIONAL DERIVATIVES**

**THM**

Suppose \( f \) is differentiable. The maximum value of the directional derivative of \( f \) at a point is \( |\nabla f| \) and it occurs when direction vector is in same dir. as \( \nabla f \).
\[ DF(x_0) = \nabla f(x_0) \cdot \overline{n} = \left| \nabla f(x_0) \right| \cos \theta \]

The constant is 1 if \( \theta = 0 \).

\[ \overline{n} = \frac{\nabla f}{|\nabla f|} \]

Consider level curves of \( f(x, y) \):

\[ f(x, y) = k \]

The gradient vector is orthogonal to the level curve.

Consider level surfaces:

\[ f(x, y, z) = k \]

At a point, \( \nabla f(x_0) \) is normal to the level surface.
WE HAVE. \( \nabla f(x_0) \) IS NORMAL TO
SURFACE \( f(x,y,z) = k \) AT \( x_0 \).
So EQN OF TANGENT PLANE AT \( x_0 \) IS
\[ \nabla f(x_0,y_0,z_0) \cdot (x-x_0,y-y_0,z-z_0) = 0 \]

(\( \nabla f(x_0) \))

Ex. GIVEN \( f(x,y,z) = xyz^2 \) FIND EQN OF
TANGENT PLANE TO LEVEL SURFACE \( f(x,y,z) = 12 \)
AS \( (3,2,1) \)
\[ \nabla f = \langle yz^2, xz^2, xy^2 \rangle \bigg|_{(3,2,1)} = \langle 4, 12, 36 \rangle \]
TANGENT PLANE IS:
\[ 4x - 12y + 36z = 72 \] OR
\[ x - 3y + 9z = 18 \]