15.5 SURFACE AREA (CONT'D)

GIVEN \( f(x,y) = z \)

\[
S.A. = \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1} \, dA
\]

EX FIND SURFACE AREA OF PART OF SURFACE \( z = 1 + 3x + 2y^2 \) ABOVE TRIANGLE W/ VERTEXES AT \((0,0); (0,1); (2,1)\)
\[ A = \iiint_D \sqrt{(f_x)^2 + (f_y)^2 + 1} \, dA \]

\[ z = 1 + 3x + 2y^2 \]

\[ \frac{\partial z}{\partial x} = 3 \]
\[ \frac{\partial z}{\partial y} = 4y \]

\[ = \iiint_D \sqrt{16y^2 + 10} \, dA \]

Type I

\[ = \int_0^2 \int_0^{1 - \frac{1}{2}x} \sqrt{16y^2 + 10} \, dy \, dx \]

Type II

\[ = \int_0^2 \int_0^{\sqrt{16y^2 + 10}} \sqrt{16y^2 + 10} \, dx \, dy \]

\[ \iiint_D \sqrt{16y^2 + 10} \, dA \]

\[ = \int_0^{2y} \int_0^{\sqrt{16y^2 + 10}} dx \, dy \]

\[ \left. \frac{\sqrt{16y^2 + 10}}{2y} \right|_0^{2y} = \sqrt{16y^2 + 10} (2y - 0) \]
\[
\int_0^1 \sqrt{16y^2 + 10} (2y) \, dy
\]

\[
\frac{1 \, dy}{32} = \frac{1}{10} \, dy
\]

\[
y = 1, \quad u = 26
\]

\[
y = 0, \quad u = 26
\]

\[
\frac{1}{16} \int_0^{26} \sqrt{u} \, du = \frac{1}{16} \left[ \frac{2u^{3/2}}{3} \right]_0^{26}
\]

\[
= \frac{1}{24} \left[ 26^{3/2} - 10^{3/2} \right] = \frac{(26\sqrt{26} - 10\sqrt{10})}{24}
\]

Ex. (15.5.12) Find surface area of part of sphere \( x^2 + y^2 + z^2 = 4 \) inside of paraboloid \( z = x^2 + y^2 \).
\( z = f(x, y) \)

\[
x^2 + y^2 + (z-2)^2 = 4
\]

\[
(z-2)^2 = 4 - x^2 - y^2
\]

\[
z = 2 + \sqrt{4 - x^2 - y^2}
\]

\[
\frac{\partial z}{\partial x} = \frac{1}{2} \left( 4 - x^2 - y^2 \right)^{1/2} \left[ -2x \right] = \frac{-2x}{2\sqrt{4 - x^2 - y^2}}
\]

\[
\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{4 - x^2 - y^2}}
\]
\[ A = \iint_D \sqrt{1 + \left( \frac{-x}{\sqrt{4-x^2-y^2}} \right)^2 + \left( \frac{-y}{\sqrt{4-x^2-y^2}} \right)^2} \, dA \]

\[ = \iint_D \sqrt{\frac{1}{4 - x^2 - y^2}} \cdot \frac{x^2}{4 - x^2 - y^2} + \frac{y^2}{4 - x^2 - y^2} \, dA \]

\[ x^2 + y^2 \leq 3 \]

\[ = \iint_D \sqrt{\frac{(4 - x^2 - y^2) + x^2 + y^2}{4 - x^2 - y^2}} \, dA \]

\[ = \iint_D \frac{2}{\sqrt{4 - x^2 - y^2}} \, dA \]

Switch to polar coordinates:

\[ x^2 + y^2 \leq 3\]

\[ r^2 \leq 3\]

\[ 0 \leq r \leq \sqrt{3}\]

\[ 0 \leq \theta \leq 2\pi\]

\[ = \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r}{\sqrt{4 - r^2}} \, dr \, d\theta \]

\[ = \int_0^{2\pi} \left[ \frac{r^2}{\sqrt{4 - r^2}} \right]_0^{\sqrt{3}} \, d\theta \]

\[ = \int_0^{2\pi} \left( \frac{3}{2} \right) \, d\theta \]

\[ = \left( \frac{3}{2} \right) \theta \bigg|_0^{2\pi} \]

\[ = \pi \]
\[ r = \sqrt{3} \]

\[ \int_0^\frac{2\pi}{3} \int_0^{\frac{2\pi}{2}} \int_0^{4-r^2} \, \mathrm{d}u \, \mathrm{d}r \, \mathrm{d}\theta = 2\pi \int_0^{\frac{2\pi}{3}} \frac{\theta \, \mathrm{d}u}{\sqrt{u}} \bigg|_1^4 \]

\[ = 2\pi \int_1^4 \frac{+ \, \mathrm{d}u}{\sqrt{u}} = 2\pi \int_1^4 \frac{\, \mathrm{d}u}{u^{1/2}} \bigg|_1^4 \]

\[ = 4\pi \left[ (4)^{1/2} - (1)^{1/2} \right] = \frac{4\pi}{2} = \frac{2\pi}{1} \text{ SQ. UNITS} \]

**15.6 TRIPLE INTEGRALS**

Box: \([a, b] \times [c, d] \times [e, f] \]

\[ \Delta V = \Delta x \Delta y \Delta z \]

\[ f(x, y, z) \]

\[ \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{p} f(x_{ijk}, y_{ijk}, z_{ijk}) \Delta V \]
Triple integral of \( f(x,y,z) \) over box \( B \),

\[
\iiint_B f(x,y,z) \, dV = \int_a^b \int_c^d \int_e^f f(x,y,z) \, dx \, dy \, dz
\]

Total of 6 orders of integration:

\[
\begin{align*}
&dx \, dy \, dz \\
&dx \, dz \, dy \\
&dx \, dy \, dz \\
&dy \, dx \, dz \\
&dy \, dz \, dx \\
&dz \, dx \, dy \\
&dz \, dy \, dx
\end{align*}
\]