15.6 Triple Integrals

Suppose a function of three variables is defined on a 'box': \( B = [a, b] \times [c, d] \times [r, s] \)

Definition: The triple integral of \( f \) over the box \( B \) is

\[
\int \int \int_B f(x, y, z) \, dV = \lim_{l,m,n \to \infty} \sum \sum \sum f(x_{ijk}, y_{ijk}, z_{ijk}) \Delta V
\]

Theorem (Fubini): \( \int \int \int_B f(x, y, z) \, dV = \int_r^s \int_c^d \int_a^b f(x, y, z) \, dx \, dy \, dz \)

Note: there are six possible 'orders' of integration.

Triple integrals over general regions \( E \subseteq \mathbb{R}^3 \)

Definition: A region \( E \) is of type I in case

\[
E = \{(x, y, z) \mid a \leq x \leq b; c \leq y \leq d, \ u_1(x, y) \leq z \leq u_2(x, y) \}
\]

If \( E \) is type I, \( \int \int \int_B f(x, y, z) \, dV = \int \int_D \left( \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \right) \, dA \),

where \( D \) is the 'shadow' of \( E \) on the \( xy \)-plane.

example: Evaluate \( \int \int \int_E y \, dV \), where \( E \) is bounded by the planes \( x = 0, y = 0, z = 0, \) and \( 3x + 2y + z = 6 \).

Setting up triple integrals with different orders of integration:

example (15.6.30): Express \( \int \int \int_E f(x, y, z) \, dV \), where \( E \) is the solid bounded by \( y^2 + z^2 = 9, \ x = -2, \) and \( x = 2 \).

example (15.5.34): \( \int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z) \, dy \, dz \, dx \). Rewrite as an equivalent integral in other orders (figure in text).

Note: \( \int \int \int_E (1) \, dV = \text{volume of } E \)