15.3 Double Integrals in Polar Coordinates

Note: Polar coordinates are discussed in section 10.3.

Conversions: \((r, \theta) \rightarrow (x, y)\) \quad x = r \cos \theta \quad y = r \sin \theta

\((x, y) \rightarrow (r, \theta)\) \quad r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x}

\[\int \int_R f(x, y) \, dA\] Suppose that \(R\) is better described in polar coordinates.

example: \(1 \leq x^2 + y^2 \leq 4, \ y \geq 0\) \quad \rightarrow \quad 1 \leq r \leq 2, \ 0 \leq \theta \leq \pi

Recall: area of a sector: \(A_{slice} = \frac{1}{2} r^2 (\Delta \theta)\)

Polar 'rectangle': \(a \leq r \leq b, \ \alpha \leq \theta \leq \beta\)

\[\Delta A = \frac{1}{2} R^2 \Delta \theta - \frac{1}{2} r^2 \Delta \theta = \frac{1}{2} (R^2 - r^2) \Delta \theta = \frac{1}{2} \left[ (R + r)(R - r) \right] \Delta \theta\]

\[= \frac{R+r}{2} \left( \Delta r \right) \Delta \theta = r^* (\Delta r) \Delta \theta\] at limits, becomes \(dA = r \, dr \, d\theta\)

Change to polar coordinates: If \(f\) is continuous on a polar rectangle \(R\) given by

\[0 \leq a \leq r \leq b, \ \alpha \leq \theta \leq \beta, \ \text{then}\]

\[\int \int_R f(x, y) \, dA = \int_a^b \int_{\alpha}^{\beta} f(r \cos \theta, r \sin \theta) (r \, dr \, d\theta)\]

example: Evaluate \(\int \int_R y \, dA\), where \(R\) is the region in the first quadrant

bounded by \(x^2 + y^2 = 9; \ y = x; \ \text{and} \ x = 0.\)

example (15.3.13) Evaluate \(\int \int_R \arctan \left( \frac{y}{x} \right) \, dA\),

where \(R = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4; \ 0 \leq y \leq x\}\)

example (15.3.32) Evaluate \(\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2} \, dy \, dx\)