Chapter 15 - Multiple Integrals

15.1 Double Integrals over Rectangles

Recall: Area \( \approx \sum f(x_i^*) \Delta x_i \)

Given \( f(x, y) \) defined on the rectangle \( R = [a, b] \times [c, d] \)

Divide \( R \) into "subrectangles"

\( R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \)

In each rectangle, choose a 'sample point' \( (x_{ij}^*, y_{ij}^*) \)

The volume of each section over a subrectangle can be approximated by the volume of a 'box':

\( V_{box} = f(x_{ij}^*, y_{ij}^*) \cdot \Delta A \)

So \( V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \cdot \Delta A \) called a double Riemann sum.

Definition: The double integral of \( f \) over the rectangle \( R \) is

\[
\iint_R f(x, y) \, dA = \lim_{m,n \to \infty} \left( \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \cdot \Delta A \right)
\]

example: (15.1.1) Estimate the volume of the solid that lies below the surface \( z = xy \) and above the rectangle \( R = [0, 6] \times [0, 4] \). Use a double Riemann sum with \( m = 3, n = 2 \) and choose the sample points to be:

a. lower right corners.

b. midpoints.

example (15.1.8) (overhead) Estimate \( \iint_R f(x, y) \, dA \) with the given contour map, where \( R = [0, 1] \times [0, 1] \).
15.1 Double Integrals over Rectangles (p. 2)

Iterated Integrals

Suppose that \( f(x, y) \) is continuous on \( R = [a, b] \times [c, d] \)

Notation: \( A(x) = \int_c^d f(x, y) \, dy \) = 'partial' integral w.r.t. \( y \) = a function of \( x \)

so \( \int_a^b A(x) \, dx = \int_a^b \left( \int_c^d f(x, y) \, dy \right) \, dx \)

Definition: An \textbf{iterated integral} is

\[
\int_a^b \int_c^d f(x, y) \, dy \, dx = \int_a^b \left( \int_c^d f(x, y) \, dy \right) \, dx \quad \text{[or other order]}
\]

element: \( \int_0^2 \int_1^4 (x + \sqrt{y}) \, dy \, dx \) \ [do both ways]

\textbf{Theorem} (Fubini-1907): If \( f \) is continuous on \( R = [a, b] \times [c, d] \) , then

\[
\int \int_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy
\]

Note that Fubini's Theorem works if \( f \) is a bounded function, possibly discontinuous on only a finite number of smooth curves.

example (15.2.8): Evaluate: \( \int_1^3 \int_1^5 \frac{ln \, y}{xy} \, dy \, dx \)

element (15.2.20) Evaluate: \( \int \int_R \frac{x}{1+xy} \, dA \), where \( R = [0, 1] \times [0, 1] \)

Note: 'separable' iterated integrals are of the form

\[
\int \int_R f(x, y) \, dA = \int_a^b \int_c^d \left[ g(x)h(y) \right] \, dy \, dx = \int_a^b \left( \int_c^d g(x)h(y) \, dy \right) \, dx
\]

\[
= \int_a^b g(x) \left( \int_c^d h(y) \, dy \right) \, dx = \left( \int_c^d h(y) \, dy \right) \int_a^b g(x) \, dx
\]