14.2 Limits and Continuity

example: Compare \( f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \) and \( g(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \) 'near' \((0, 0)\)

Definition: The **limit** of \( f(x, y) \) as \((x, y) \to (0, 0)\) is \( L \), and denoted

\[
\lim_{(x,y) \to (0,0)} f(x, y) = L \quad \text{if} \quad f(x, y) \text{ is 'close' to } L \text{ whenever } (x, y) \text{ is 'sufficiently close' to } (a, b). \quad \text{[That is, given } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that when }
\sqrt{(x-a)^2 + (y-b)^2} < \delta, \text{ then } |f(x, y) - L| < \epsilon.] \]

Note that if along one curve \( C_1 \), \( f(x, y) \to L_1 \) and along another curve \( C_2 \), \( f(x, y) \to L_2 \neq L_1 \), then \( \lim_{(x,y) \to (0,0)} f(x, y) \) does not exist.

example: \( \lim_{(x,y) \to (0,0)} \left( \frac{x^2 - y^2}{x^2 + y^2} \right) \) approach along \( y\)-axis: \((x, 0) \to (0, 0)\)

Limit is 1.

approach along \( x\)-axis: \((0, y) \to (0, 0)\)

Limit is \(-1\).

Hence limit DNE.

example: \( \lim_{(x,y) \to (0,0)} \left( \frac{xy}{x^2 + y^2} \right) \) approach along \( y = mx \) for different \( m \).

example: \( \lim_{(x,y) \to (0,0)} \left( \frac{x^2 y}{x^2 + y^2} \right) \)

1) approach along \( x\)- and \( y\)-axes, along any line \( y = mx \). Limit is 0.

2) approach along curve \( y = x^2 \). Limit is \( \frac{1}{2} \)

Hence, limit DNE.

Definition: A function \( f(x, y) \) is **continuous** at \((a, b)\) if \( \lim_{(x,y) \to (0,0)} f(x, y) = f(a, b) \).

\( f \) is continuous on a region \( D \) if \( f \) is continuous at \((a, b)\) for all points \((a, b) \in D\).

Note: can use polar coordinates for evaluate limits:

\[
\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{r \to 0} f(r \cos \theta, r \sin \theta) = L \quad \text{iff} \quad L \text{ is independent of } \theta.
\]