13.3 Arc Length and Curvature

Recall: Arclength \( L = \int ds = \int_{\text{start}}^{\text{end}} \sqrt{(dx)^2 + (dy)^2} = \ldots \)

Arclength: \( L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt = \int_{a}^{b} |r'(t)| \, dt \)

equation: Find the length of curve given by \( r(t) = \langle t^2, 2t, \ln t \rangle, \ 1 \leq t \leq e \)

Note that a curve \( C \) could have multiple parametric representations: for example, \( <t, t^2> \) for \( t > 0 \), \( <t^2, t^4> \) for \( t \neq 0 \), \( <e^t, e^{2t}> \) yield same curve.

Want new parameter, independent of \( t \); parametrize a curve w.r.t. arc length \( s \).

Arclength function \( s(t) = \int_{a}^{t} |r'(u)| \, du = \int_{a}^{t} \sqrt{(f')^2 + (g')^2 + (h')^2} \, du \)

Note that by F.T.C., \( s'(t) = |r'(t)| \)

Once we find this integral, we solve for \( t \) in terms of \( s \) (if possible):
\[ r(t) = r(t(s)) \]

equation: \( r(t) = \langle 1 + 2t, 3 + t, -5t \rangle, \ t \geq 0 \)

Definition: The curvature of a (smooth) curve is \( \kappa = \left| \frac{dT}{ds} \right| \); this measures the rate of change of the unit tangent vector \( T \) with respect to \( s \). Note that the magnitude of \( T \) doesn't change just the direction, so \( \kappa \) is independent of \( t \).

Easy calculation for curvature: \( \kappa = \left| \frac{dT}{ds} \right| = \left| \frac{dT}{dt} \right| \frac{dt}{ds} = \left| \frac{T'(t)}{|r'(t)|} \right| \)

equation: Find the curvature of the circle: \( x = a \cos t \)
\[ y = a \sin t \]
\[ \kappa = \frac{1}{a} \]
13.3 Arc Length and Curvature (continued)

example: Given \( \mathbf{r}(t) = \langle t, t^2, t^3 \rangle \), find \( \kappa \).

Theorem: If a curve \( C \) in \( \mathbb{R}^2 \) is given by \( y = f(x) \) [so \( \mathbf{r} = \langle x, f(x) \rangle \)],

\[
\kappa = \frac{|f''(x)|}{\left(1+\left[f'(x)\right]^2\right)^{3/2}}
\]

Normal and Binormal vectors

Since \( |T(t)| = 1 \), \( T'(t) \) is orthogonal to \( T(t) \)

Definition: the unit normal vector \( N(t) = \frac{T'(t)}{|T'(t)|} \).

The (unit) binormal vector is \( B(t) = T(t) \times N(t) \)

The normal plane (for a curve \( C \)) at a point \( P \) is the plane formed by \( B \) and \( N \). The normal vector to the normal plane is therefore \( T \).

The osculating plane (for a curve \( C \)) at a point \( P \) is the plane formed by \( T \) and \( N \). It is the plane that comes closest to containing the part of the curve \( C \) near the point \( P \).

The osculating circle (or the circle of curvature) is the circle that lies in the osculating plane that best matches the curvature of \( C \) at \( P \); the osculating circle shares the same tangent, normal and curvature as the curve \( C \) at \( P \). Note that the radius of the osculating circle is \( \rho = \frac{1}{\kappa} \).

Note that \( T, N, \) and \( B \) form a 'moving' coordinate system for an object 'traveling' along the curve \( C \). The tangent vector \( T \) points in the direction of travel, the normal vector \( N \) points in the direction of any 'turns', and the binormal vector \( B \) is cross product of \( T \) and \( N \).